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Anomalous Behavior of the Vertical Structure of Rossby Waves on Non-Zonal Shear Flow in the Vicinity of the Focus

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Abstract
Purpose. The work aims to study the behavior of vertical barotropic-baroclinic modes of Rossby waves on a non-zonal shear flow in the vicinity of the focus.
Methods and Results. Inferred from the reference equation, we consider some variants of the behavior of eigenfunctions in the vicinity of the focus. It is shown that the number of possible variants for non-zonal flows increases compared with the zonal case. This means that qualitatively new additional scenarios appear in the case of a non-zonal flow compared with the problem for internal waves when the behavior of Rossby waves in the vicinity of the localization level qualitatively coincides with the behavior for the zonal case, herewith the coefficient of a passage through the focus is always exponentially small. The solution becomes extremely sensitive to the initial parameters of the wave incident on the non-zonal focus. Another important point is that the second, additional anomalous focus appears on the non-zonal flow. When a wave falls on this focus on one side, it behaves like a classic focus with a classic wave adhering. And when falling from the opposite side, the Rossby wave does not notice the focus and passes it without a short-wave transformation. In the problem, abnormal scenarios appear with the passage of the focus without difficulty with a coefficient of passage equal to one in addition to the scenario with an infinitely long time adhering to the focus and an exponentially small coefficient of passage.
Conclusions. The anomalous behavior of Rossby waves in the horizontal plane on non-zonal flows is accompanied by anomalous behavior of the vertical mode, in contrast to the strictly zonal case of flow with different kinematics. The eigenvalues of the Sturm-Liouville problem change abruptly during the transition from the non-zonal to the zonal case. As a consequence, there is no limit transition from a weakly non-zonal case to a strictly zonal one. Such an extremely ambiguous analytical behavior of Rossby waves in the vicinity of the focus on baroclinic non-zonal flows rather indicates the absence of analytical prediction and the need for a deeper and more detailed analysis using numerical methods.

Keywords: Rossby waves, non-zonal flow, focus, Sturm-Liouville problem

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Introduction

The problem of analyzing the interaction of shear flows and the waves generated by them is now extremely relevant for understanding the synoptic variability of the ocean and has been studied both in the linear formulation for internal gravity waves and Rossby waves [1–4] and in the nonlinear formulation [5–8]. An overwhelming number of papers in geophysics, which appeared in recent years, were related to numerical methods. At the same time, there were extremely few analytical works, whose authors tried to understand the main issues of wave and current interaction on a qualitative level. From an applied point of view the following question is extremely important: can the solutions known in theoretical physics be projected onto Rossby waves in the ocean?

It is known that mathematics, as a rule, “works” perfectly well for the case of a purely zonal flow; however, any, even weak, deviation from zonality makes the solutions extremely capricious, and often many theorems basically stop working [9–11].

Nowadays, the current progress of modern earth remote sensing methods, particularly the advances in satellite altimetry and the development of software packages for automatic identification of ocean eddies, makes the problem of Rossby eigenvalues on barotropic-baroclinic flows extremely relevant. One of the methods for studying the dynamics of ocean waves is the “vertical modes – horizontal rays” method [12]. Since the horizontal scales of Rossby waves make up tens to hundreds of kilometers, this approximation works well in the open ocean. If one accepts stratification as constant and does not consider topography and baroclinic background flows, then the vertical mode of Rossby waves is determined by one stratification and does not depend on the β-parameter. In this regard, the Rossby wave becomes similar to an ordinary internal wave and its vertical mode is an ordinary trigonometric function with a classical quantization of eigenvalues of the Sturm – Liouville problem. In this formulation, the determining factor is the horizontal inhomogeneity of the large-scale flow. Horizontal flow variations are the leaders in the problem. However, along with the similarity of the problem formulation for internal waves and Rossby waves, there are both qualitative and quantitative differences.

The first difference between Rossby waves and internal waves is that for Rossby waves there are two qualitatively different scenarios of wave ray evolution, which is a consequence of the β-parameter presence in the problem for both the zonal background flow [13] and the non-zonal flow [14]. For a non-zonal case, a qualitatively new scenario emerges, which is associated with the phenomenon of *overshooting*, i.e. a Rossby wave ducks under the critical layer. Another scenario is *adhering*, where a Rossby wave asymptotically approaches a critical layer [3].

The second and most significant difference between internal waves and Rossby waves is as follows. For the internal waves, an addition of the background flux baroclinity does not qualitatively change the scenario of the wave train evolution. An infinite countable spectrum of the Sturm – Liouville boundary value problem with a trigonometric set of eigenfunctions smoothly passes into a new infinite countable spectrum, but with the eigenfunctions in the form of exponentially-majorized Hermite polynomials. This introduces phenomena such as vertical focusing and “non-dispersive” focusing [15]. However, the focal point for internal waves still remains a kind of a “black hole”, with the rays being the “leaders” and the vertical modes being the “ardent followers” with some secondary role.

The mathematical concepts of “focus” and “focusing” from the theory of differential equations as applied to the problems of Rossby waves interacting with flows are thoroughly discussed in [10], and they have a certain physical meaning for the ocean. If in optics the focus is a kind of metal paraboloid that focuses a beam of rays to a point, then in the ocean such a “metal paraboloid” is inhomogeneity of the background flow velocity fields. Metaphorically speaking, the ocean has “ears”. In physics, it is close to an acoustic waveguide, but the waveguide there is homogeneous and infinite, and here it is as if gradually narrowed from finite size to a point, thus, the wave is also compressed vertically to a point, i.e. focused. Focus, therefore, is the vertical contraction of the wave to a point size at some vertical horizon, and in the horizontal plane, it is a regular plane wave.

It is known that the strongest wave processes are observed in the vicinity of frontal formations. For the open ocean, the horizontal gradients of the background flows are much weaker than the vertical ones, and then the analytical approach in the WKB approximation, where the slowness of the horizontal gradients compared to the vertical variability is a small parameter, looks quite justified. In contrast, for a region of strong shear flows, horizontal and vertical gradients of background flows must be considered simultaneously. The analytical method that simultaneously accounts for these gradients in this kind of problem is the construction of a reference two-dimensional equation [10].

In practice [16–18], in order to calculate phase velocities of Rossby waves, a fundamentally one-dimensional problem is solved, which takes into account only the vertical profile of the velocity field $U(z)$. It is important to note that the idea of vertical focusing of Rossby waves, which was originally formulated purely analytically [13], was subsequently confirmed in practical numerical calculations for real ocean currents [17, 18]. However, the question “How much will the spectral problem change if the simultaneous influence of both vertical and horizontal gradients of the background flow field is taken into account?” has not yet been fully studied even analytically. For zonal flows, analytical calculations indicate that such focusing occurs at least for the open ocean [13]. For a non-zonal flow, the results are known when non-zonality leads to extremely incomprehensible and unexpected effects when an eddy “drives” into its own beta plume [6, 7]. We shall note that all these conclusions were obtained in the “shallow water” approximation. In our setting, we construct a reference equation in the quasi-geostrophic approximation, for which a Galilean invariance [11] takes place. We consider an analytical two-dimensional model of vertical focusing that takes into account non-zonal background flow and show that vertical focusing is an extremely strong phenomenon, while non-zonality manifests itself in terms of first derivatives in the model equation and does not affect the highest second derivative in any way. However, the solution in the vicinity of the non-zonal critical layer is not unique. This is due to the *overshooting* phenomenon when a wave can cross the critical layer, and in this case, the solution in the vicinity of the critical layer is no longer a focus but is described by a certain constant. Thus, the purpose of this work is to study the behavior of vertical barotropic-baroclinic modes of Rossby waves on a non-zonal flow in the vicinity of the focus.

Problem statement

Let us consider the equation of vorticity on the β-plane linearized against a plane-parallel shear flow U directed at some fixed angle θ to the parallel [3]:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left[\nabla_h^2\Psi_A(x,y,z,t)+\left(\frac{1}{S}\Psi_A(x,y,z,t)_z\right)_z\right]-$$
$$-\frac{\partial\Psi_A(x,y,z,t)}{\partial y}\beta\sin\theta+\frac{\partial\Psi_A(x,y,z,t)}{\partial x}\left[\beta\cos\theta-U_{yy}-\left(\frac{1}{S}U_z\right)_z\right]=0, \quad (1)$$

where $\Psi_A(x,y,z,t)$ is the flow function (pressure); $S=N^2/f^2$, N is the Brunt – Väisalä frequency, f is the Coriolis parameter; $\beta=\frac{df}{dy}$. The coordinate system (x,y,z) is right-handed, t is time; axis x is directed along the flow at an angle θ to the parallel (Fig. 1). To analyze this equation, we use the wave approach.

Since the background flow is homogeneous along the longitudinal coordinate x and independent of t, the solution for perturbation will have the following form:

$$\Psi_A(x,y,z,t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Psi(k,y,z,\omega)\exp\left[i(kx-\omega t)\right]dk\,d\omega. \quad (2)$$

Here k is the wavenumber directed towards the x-axis; ω is the wave frequency. Substituting (2) into (1), we obtain the following equation for the function $\Psi(k,y,z,\omega)$:

$$(\omega-kU)\left[\frac{\partial^2}{\partial y^2}-k^2+\left(S^{-1}\frac{\partial}{\partial z}\right)_z\right]\Psi-i\frac{\partial\Psi}{\partial y}\beta\sin\theta-k\Psi\left[\beta\cos\theta-U_{yy}-\left(S^{-1}U_z\right)_z\right]=0. \quad (3)$$

The key point of this paper is a non-trivial nonlinear variable substitution that allows one to perform a variable separation in a two-dimensional inhomogeneous equation with initially non-separating variables. Such an approach was presented in [10], but it did not explain how to justify such a substitution. Below we will show that this approach is due to the application of WKB approximation, after which it becomes obvious how variables have to be transformed to see the automodel of the solution.

One-dimensional reference equation. Barotropic case

In the shallow water approximation, the solution can be searched for using the split-variable method:

$$\Psi_2(k,y,z,\omega)=\Phi(k,y,\omega)\sum_{n=0}^{\infty}\cos\left(\frac{\pi n}{H}z\right), \quad (4)$$

where H is the ocean depth. For the function $\Phi(k,y,\omega)$ from (3) for the linear velocity profile of the background flow, we obtain the following linear one-dimensional equation:

$$(\omega-kU)\left[\frac{\partial^2}{\partial y^2}-k^2-S^{-1}m^2\right]\Phi-i\frac{\partial\Phi}{\partial y}\beta\sin\theta-\Phi k\,\beta\cos\theta=0, \quad (5)$$

where $m = \pi n / H$, $S = N^2 / f^2$, N is the Brunt – Väisalä frequency, which is considered to be constant, and H is the ocean depth.

Let us consider the behaviour of the solution in the vicinity of the critical layer y_c. We shall make the following substitution:

$$-(\omega - kU) = kU_y y - kU_y y_c = kU_y (y - y_c). \quad (6)$$

Then we move the origin of the coordinates to the critical layer. With these assumptions from (5), we obtain the following barotropic reference (model, $m = 0$) equation in the vicinity of the critical layer:

$$y\,\Phi_{yy} + a\,\Phi_y + b\,\Phi = 0. \quad (7)$$

Here

$$a = i a_0, \quad a_0 = \frac{\beta \sin\theta}{kU_y}, \quad b = \frac{\beta\cos\theta}{U_y}. \quad (8)$$

The solution of equation (7) has a branching point at zero. The traditional approach to the analysis of the solution of equation (7) can be done in terms of the Bessel function. However, for the sake of consistency of the description and preservation of the unified approach, we find the solution of equation (7) in the form of the Fourier integral and construct its asymptotics independently, without involving the apparatus of special functions.

Therefore, we are looking for a solution in the following form:

$$\Phi(k, y, \omega) = \int_{-\infty}^{+\infty} G(k, l, \omega) \exp(i l y)\, dl. \quad (9)$$

Using the properties of the Fourier transformation (which are derived from the definition of the Fourier transformation and are formally obtained from (9) by differentiating by y as a parameter), we obtain

$$\Phi \to G, \quad \Phi_y \to i l G, \quad \Phi_{yy} \to -l^2 G, \quad y\Phi_{yy} \to -i\left(l^2 G\right)_l, \quad (10)$$

where the arrow denotes the corresponding Fourier transformation; i is the imaginary unit. We obtain the following equation for the Fourier-transformed image G:

$$-i l^2 G_l - 2 i l G + (i a l + b) G = 0. \quad (11)$$

Integrating (11) and substituting into (9), we find the desired general solution in the form of a Fourier integral. Since the integrand contains a term containing the factor $\ln|l|$, the integration limits $(-\infty, +\infty)$ must be divided into two intervals $(-\infty, 0)$, $(0, +\infty)$. Thus, the general solution will be represented as the sum of the solutions to the right and left of the singular point. Further, formally, we will have to sew these solutions together. At first, we are to analyze one part of the solution, on one side of the critical layer, and find out the behavior of the solution at infinity and in the vicinity of the critical layer without passing through the singular point. Thus, below we restrict ourselves to considering only one of the solutions:

$$\Phi\left(k,y,\omega\right)=A\left(k,\omega\right)\int\limits_{0}^{+\infty} l^{-2}\exp\left(i\left(a_0\ln l+\frac{b}{l}+l\,y\right)\right)d\,l\ . \qquad (12)$$

To analyze the integral (12), we assume that the solution is localized in some region of physical space and attenuates at infinity. Then the Fourier-transformed image of our solution will also be localized in the vicinity of some wavenumber l_0 in the phase l-space. In this case, the width of the localization region Δl in the phase space must be less than the central wave number l_0, so that the contribution from the power singularity in the integrand (12) does not contribute to the asymptotics of the general solution:

$$\Delta l << l_0\ . \qquad (13)$$

We introduce the definition of the phase of solution (12) as follows:

$$\varphi\left(k,l,y,\omega\right)=a_0\ln l+\frac{b}{l}+l\,y\ . \qquad (14)$$

Then the equation for the stationary point of the phase $\left(l=l_c\right)$ has the following form

$$\varphi\left(k,l,y,\omega\right)_l=\frac{a_0}{l_c}-\frac{b}{l_c^2}+y=0\ . \qquad (15)$$

From (15) using (8) we find the expression for the y-coordinate of the quasimonochromatic wave solution (train):

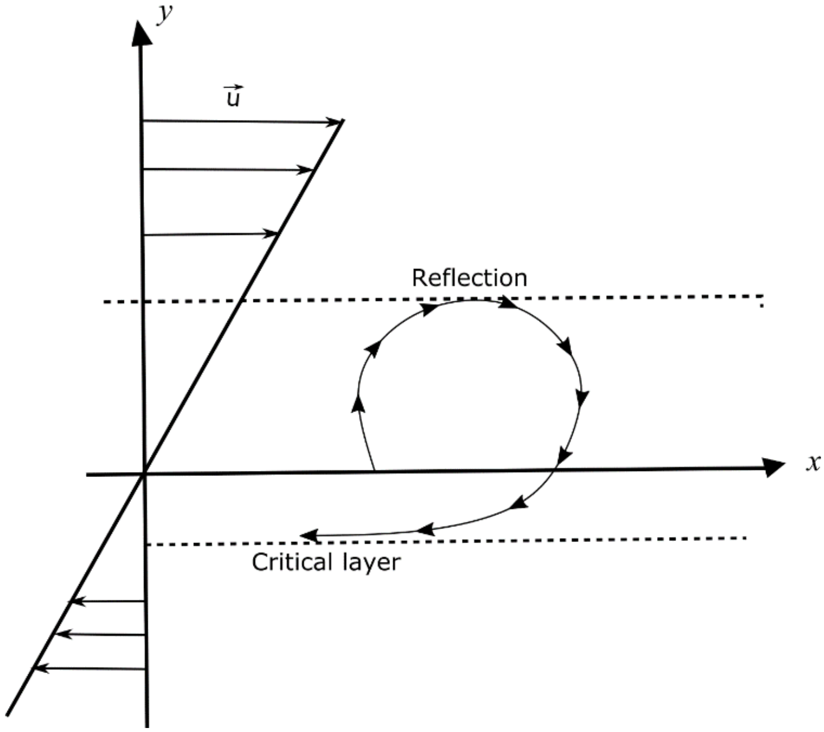
$$y=-\frac{\beta\cos\theta}{U_y l_c^2}+\frac{\beta\sin\theta}{kU_y\,l_c}\,. \qquad (16)$$

Equation (16) is quadratic with regard to $l_c=l_c(y)$. The quadratic equation will have two roots in the transparency region, one solution in the reflection point, and no solutions in the geometric shadow region. From equation (16) we find two modes when approaching the critical layer. The first mode is the passage of the critical layer without short-wave transformation when the solution is proportional to some constant in the vicinity of the critical layer. The second mode is the short-wave transformation. In the short-wave limit for non-zonal flows from (16) we obtain the following asymptotics:

$$y\sim\frac{\beta\sin\theta}{kU_y\,l_c},\quad l_c\to\infty. \qquad (17)$$

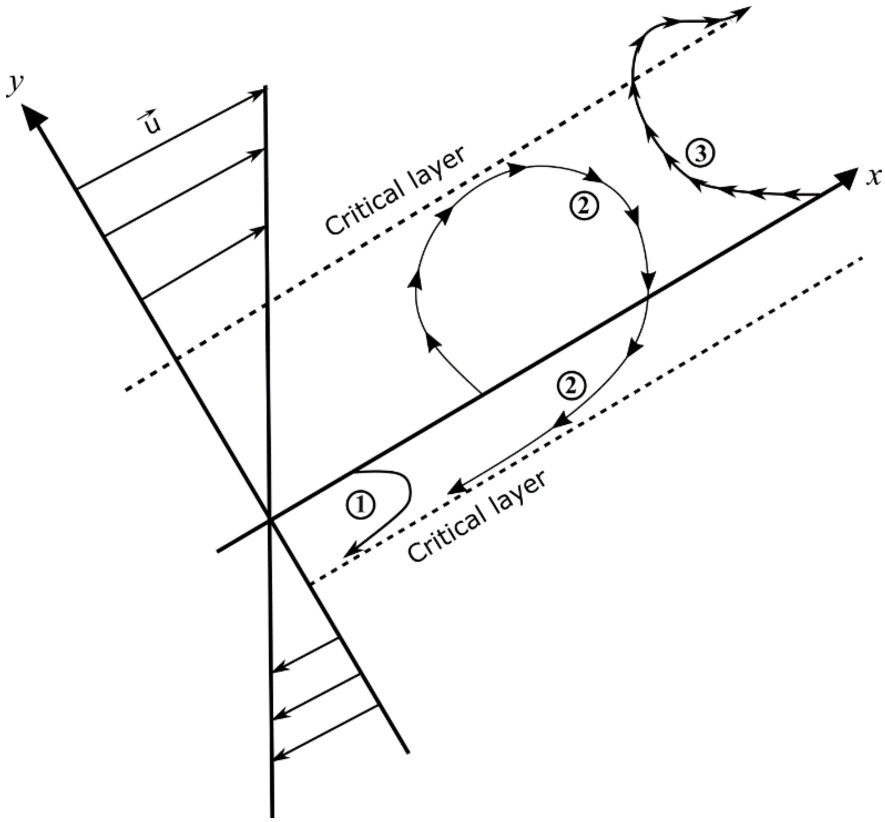
Therefore, a mathematically qualitatively new moment for the non-zonal critical layer, in addition to the focusing mode, is the second solution – a certain constant. Let us explain what this solution is from a physical point of view.

In Fig. 1 (zonal flow) the critical layer is shown by a dotted line. The approach to the critical layer is asymptotically long in time and is accompanied by vertical focusing of the mode in the vicinity of the velocity profile extremum. In Fig. 2 (non-zonal flow) there are already two critical layers. In this case, for the trajectory indicated in Fig. 2 by number 3, when approaching the critical layer, the *overshooting* takes place.



F i g. 1. Rossby waves track while interacting with a zonal flow

The wave initially crosses the critical layer without any special transformations and then, having reflected from a larger value of the velocity field, approaches the critical layer asymptotically with vertical mode focusing. It is the point of intersection of the critical layer that is the new moment for the non-zonal flow. At this point, the solution is described by constants – both the wave number and the amplitude. Such ambiguity of the solution in the vicinity of the critical layer makes the concept of “critical layer as an asymptotic regime” incorrect since the meaning of the term “asymptote” is lost. Asymptote in Greek means “never reachable”. *Overshooting* – passing the critical layer – makes this statement incorrect.



F i g. 2. Rossby wave tracks while interacting with a non-zonal flow: *1* – asymptotic approximation to the critical layer (*adhering*); *2* – asymptotic approximation to two critical layers (*double adhering*); *3* – crossing of the critical layer and asymptotic approximation to it from the opposite side (*overshooting*)

Let us note the following important circumstance. The expression for the coordinate of the center of the quasi-monochromatic packet (16) does not depend on whether *y* takes small or large values, i.e. expression (16) is valid at $y \in (0, +\infty)$. Further, by expanding phase (15) by terms (up to square terms), in the short-wavelength limit we obtain

$$\mu_y \equiv \left. \frac{\partial^2 \varphi(k, l, y, \omega)}{\partial l^2} \right|_{l=l_c} \sim -\frac{a_0}{l_c^2}. \tag{18}$$

Using the Poisson integral, we finally obtain

$$\Phi(k, y, \omega) \sim A(k, \omega) \frac{1}{\sqrt{|\mu_y|}} \exp(+i\, a_0 \ln l_c). \tag{19}$$

Substituting (17) and (18) into (19), we find the asymptotics in the short-wave limit for the wave incident on the critical layer:

$$\Phi\left(k,y,\omega\right)\sim A\left(k,\omega\right)y^{1-ia_0}\sim A\left(k,\omega\right)y\cos\left(a_0\ln y\right). \tag{20}$$

The one-dimensional solution we obtain (20) is an asymptotic law; this solution coincides with the known asymptotics in terms of special functions [19]. The constructed solution yields a two-dimensional solution as a simple multiplication of two one-dimensional solutions (see formula (4)), in which a vertical mode is already present.

Fig. 1 represents schematically the behavior of the Rossby wave in interaction with the zonal flow $\vec{U}$. It can be seen that there is a reflection point (upper point, the wave is reflected from the flow) and there is a critical layer (lower point, the wave adheres to the critical layer asymptotically long).

A two-dimensional reference equation. Baroclinic case

In this paper we modify the two-dimensional reference equation discussed earlier [10]:

$$\left(\Psi_1\right)_{zz}+\left(\frac{y}{L_y}+\frac{z^2}{L_z^2}\right)\left(\Psi_1\right)_{yy}+a\left(\Psi_1\right)_y=0. \tag{21}$$

Here $a\sim\ i\sin\theta$ is a purely imaginary value and reflects the fact of non-zonality of the flow [9, 10]. The new reference equation, despite the complexity of coefficients, withstands the procedure of constructing the solution in terms of the Fourier integral. By analogy with [10], we will consider the integral on the interval $\left(0,\ +\infty\right)$:

$$\Psi_1\left(k,y,z,\omega\right)=\int_0^{+\infty}G\left(k,l,z,\omega\right)\exp\left[+il\,y\right]dl. \tag{22}$$

We substitute the expansion (22) into equation (21) and, taking into account (10), we obtain

$$G_{zz}-\frac{l^2z^2}{L_z^2}G-i\frac{l^2}{L_y}G_y-i\frac{2\,l}{L_y}G+i\,a\,l\,G=0. \tag{23}$$

We divide (23) by l and multiply by L_z. We obtain

$$\frac{L_z}{l}G_{zz}-\frac{l\,z^2}{L_z}G-i\frac{l\,L_z}{L_y}G_y+i\left(a-\frac{2}{L_y}\right)L_z\,G=0, \tag{24}$$

where the following notations are traditionally used: L_y is the characteristic scale of variability in the y-coordinate in the vicinity of the focus; L_z is the characteristic scale of variability in the z-coordinate in the vicinity of the focus [20, 21].

We perform the following substitution of variables: $\left(l,\,z\right)\to\left(\eta,\ \varphi\right)$, where

$$\eta = \frac{z\, l^{1/2}}{L_z^{1/2}}, \quad \varphi = l\,. \tag{25}$$

In the new variables $(\eta,\ \varphi)$, equation (24) takes the form of an equation with separable variables:

$$G_{\eta\eta} - \eta^2 G - i\,\frac{\eta\, L_z}{2L_y} G_\eta - i\,\frac{\varphi\, L_z}{L_y} G_\varphi + Q\, G = 0\,. \tag{26}$$

The following notation is introduced here:

$$Q = i\left(a - \frac{2}{L_y}\right) L_z\,. \tag{27}$$

We will look for solutions using the split-variable method:

$$G(\eta,\ \varphi) = H(\eta)\ F(\varphi)\,. \tag{28}$$

Then for $H(\eta)$ we obtain the following equation:

$$H_{\eta\eta} - i\frac{\eta\, L_z}{2L_y} H_\eta - \left(\eta^2 + \mu_0^* - Q\right) H = 0\,. \tag{29}$$

Here μ_0^* is a separation constant. Further, we assume

$$\mu_0^* - Q \equiv \mu_0\,. \tag{30}$$

The term with H_η in equation (29) is eliminated by the following substitution:

$$H_\eta = P(\eta) \exp\left(i\,\frac{L_z}{8L_y}\eta^2 \right). \tag{31}$$

For $P(\eta)$ we obtain the equation

$$P_{\eta\eta} + P\left[-\eta^2 \left(1 - \frac{L_z^2}{16L_y^2} \right) - \mu_0 + i\,\frac{L_z}{4L_y} \right] = 0\,. \tag{32}$$

The asymptotic analysis of the one-dimensional vertical problem in the short-wave approximation is classical (see Appendix). We shall dwell on it in more detail. Since we are looking for solutions localized in the neighborhood of some level $z = z_0$, we see from equation (32) that the coefficient at η^2 must be positive. Consequently, we obtain the following condition for the existence of localized solutions:

$$\left(1 - \frac{L_z^2}{16\, L_y^2} \right) > 0 \Leftrightarrow 0 < \left|L_z\right| < 4\left|L_y\right|\,. \tag{33}$$

Condition (33) reveals that the branches of the parabola, which limits the inner region of transparency from the outer region of the shadow, must be practically parallel to each other. Otherwise, a vertical mode will not be formed and the wave will not approach the critical point indefinitely. Hence, if condition (33) is not fulfilled, a reflection mode from the critical layer [10] will take place.

From equation (32) we determine the eigenvalues of the partition variable μ_0:

$$-(2m+1)=\left(\mu_0-i\frac{L_z}{4\,L_y}\right)/\left(1-\frac{L_z^2}{16\,L_y^2}\right),\quad m=0,1,2,\dots. \tag{34}$$

From (34) we obtain the eigenvalues

$$\mu_0=\frac{L_z}{Ly}\left[\frac{1}{4}i-\frac{\delta}{2}\left(m+\frac{1}{2}\right)\right],\quad \delta\equiv\left(\frac{16\,L_y^2}{L_z^2}-1\right)^{1/2},\quad m=0,1,2,\dots \tag{35}$$

and eigenfunctions

$$P(\eta)=\left[\sum_{m=0}^{\infty}H_m\left(\eta\left(1-\frac{L_z^2}{16\,L_y^2}\right)^{1/4}\right)\right]\exp\left[-\frac{\eta^2}{2}\left(1-\frac{L_z^2}{16\,L_y^2}\right)^{1/2}\right],\quad m=0,1,2,\dots, \tag{36}$$

where H_m is the Hermite polynomial.

We shall proceed to the definition of $F(\varphi)$ – the second factor in the solution (28). From (26) we obtain the equation

$$-i\frac{\varphi L_z}{L_y}F_\varphi+\mu_0^* F=0. \tag{37}$$

The solution of equation (37) has the following form:

$$F(\varphi)=\varphi^\mu,\quad \mu\equiv -i\,\mu_0^*\frac{L_y}{L_z}. \tag{38}$$

We are going to describe the parameter μ in detail:

$$\mu\equiv -i\,\mu_0^*\frac{L_y}{L_z}=-i\left(\mu_0+Q\right)\frac{L_y}{L_z}=\left[-i\frac{\beta\sin\theta}{kU_y}L_y-2\right]+\left[\frac{1}{4}+i\frac{\delta}{2}\left(m+\frac{1}{2}\right)\right]. \tag{39}$$

From (39) it can be seen that the eigenvalues consist of two parts. The first part $\left[-i\frac{\beta\sin\theta}{kU_y}L_y-2\right]$ is barotropic, it coincides with the phase of the barotropic problem (12). The second part $\left[\frac{1}{4}+i\frac{\delta}{2}\left(m+\frac{1}{2}\right)\right]$ is baroclinic. We finally obtain the following eigenvalues:

$$\mu = -\frac{7}{4} + i\left[-\frac{\beta\sin\theta}{kU_y}\ L_y + \frac{\delta}{2}\left(m+\frac{1}{2}\right)\right]. \tag{40}$$

Substituting all the found parts of the solution into the original integral (22), we find the eigenfunctions:

$$\Psi_1(k,y,z,\omega) = A(k,\omega)\int_0^{+\infty}\sum_{m=0}^{\infty} l^{\mu}\left[\sum_{m=0}^{\infty} H_m\left(\frac{z\,l^{1/2}}{L_z}\left(1-\frac{L_z^2}{16\,L_y^2}\right)^{1/4}\right)\right]\times$$
$$\times\exp\left[\frac{z^{\,2}l}{2L_z}\left(1-\frac{L_z^2}{16\,L_y^2}\right)^{1/2}\right]\exp\left[+i\,l\left(y+\frac{z^2}{8L_y}\right)\right]d\,l. \tag{41}$$

Further, the obtained eigenfunctions (41) can be reduced using simple transformations to a degenerate hypergeometric function of some complex argument. However, for finding the asymptotics of the eigenfunctions, it is the integral notation (41) that is preferable. Despite the fact that the constructed eigenfunctions (41) are the functions of two physical variables (z and y), the integral expressing them is one-dimensional. Therefore, we will use the stationary phase method again.

We rewrite the imaginary part of the integral (41) as follows:

$$\exp\left[+i\,l\left(y+\frac{z^2}{8\,L_y}\right)+i\left(-\frac{\beta\sin\theta}{kU_y}L_y+\frac{\delta}{2}\left(m+\frac{1}{2}\right)\right)\ln l\right]. \tag{42}$$

Differentiating (42) by variable l and equating it to zero, we obtain the equation for the stationary point l_c (analogous to the dispersion relation):

$$y+\frac{z^2}{8L_y} = \frac{\frac{\beta\sin\theta}{kU_y}L_y - \frac{\delta}{2}\left(m+\frac{1}{2}\right)}{l_c}. \tag{43}$$

From (43) we also obtain the following constraint on the number of modes that determine the vertical structure of the solution in the vicinity of the singular point:

$$\left(m+\frac{1}{2}\right) < 2\frac{\left|\frac{\beta\sin\theta}{kU_y}L_y\right|}{\delta}. \tag{44}$$

In doing so, as in the WKB approximation, it may turn out that such modes do not exist at all.

If we rewrite relation (43) as follows

$$l_c = \frac{\frac{\beta \sin\theta}{kU_y} L_y - \frac{\delta}{2}\left(m + \frac{1}{2}\right)}{y + \frac{z^2}{8L_y}}, \tag{45}$$

it is not difficult to demonstrate that for a non-zonal flow the asymptotics of the eigenfunctions in the vicinity of the critical point will have the following form:

$$\Psi_1(k, y, z, \omega) = A(k, \omega) \sum_{m=0}^{\infty} l_c^{\mu - 1} \left[H_m \left(\frac{z\, l_c^{1/2}}{L_z} \left(1 - \frac{L_z^2}{16\, L_y^2} \right)^{1/4} \right) \right] \exp\left[-\frac{z^{\,2} l_c}{2L_z} \left(1 - \frac{L_z^2}{16\, L_y^2} \right)^{1/2} \right]. \tag{46}$$

Analyzing the asymptotics (46), we can say that the constructed solutions are not functions of (*z*, *y*) variables, but of some curvilinear variables, which have the following form:

$$(y, z) \to \left(\left(y + \frac{z^2}{8L_y} \right), \frac{z}{\left(y + \frac{z^2}{8L_y} \right)} \right). \tag{47}$$

However, this approach using curvilinear variables was also used when solving in the WKB approximation, in which formally the following substitution of variables took place: $(y, z) \to \left(y, \frac{z}{y} \right)$. Therefore, the asymptotics of one-dimensional integrals, by and large, do not give any qualitatively new results different from the WKB solutions, except for condition (33).

The behavior of Rossby waves in interaction with a non-zonal flow $\vec{U}$ is schematically demonstrated in Fig. 2. In this case, there is no longer one but two critical layers and the nature of the Rossby wave tracks is very diverse. In addition to wave propagation mode 1 (*adhering*), there is mode 2 (*double adhering*) as well as mode 3 (*overshooting*). These modes are discussed in [3, 4, 22, 23].

Discussion and conclusions

The classical two-dimensional reference equation describing the transformation of the solution in the vicinity of the focus has already been considered earlier [10, 20, 21]. The novelty of this work lies in the fact that this equation is generalized to the case of a non-zonal barotropic-baroclinic flow. A new complex term appears in the modified reference equation, and a nonlinear change of variables allows the problem to be reduced to an equation with separable variables. The mathematical analysis developed in [9, 10] makes it possible to take into account the features associated with the appearance of this term. However, another feature arises: the eigenvalues of the problem for the non-zonal flow, although structurally similar to the zonal case (the sum of the barotropic and baroclinic components), no longer have a limiting transition to the zonal case as the background flow inclination tends to zero.

It is known that on non-zonal flows the kinematics of Rossby waves has qualitative differences from the kinematics for strictly zonal flows [9, 10]. The main difference is that the non-zonal shear flow has two critical layers instead of one, as in the zonal flow, and in the vicinity of the first focus all mathematical calculations obtained in [10] are valid and the nonlinear substitution of variables allows one to find the corresponding spectral characteristics. As a result, the obtained eigenvalues are, as before, the sum of the barotropic and baroclinic problem but the principal point is that when the flow slope angle tends to zero, the eigenvalue spectrum of the non-zonal problem no longer tends to the spectrum for the zonal case and there is a jump-like behavior of eigenvalues.

The second critical layer on a non-zonal flow lies in the transparency region and has a nontrivial kinematics in the form of *overshooting* [9, 10]. In this paper, we show that such nontrivial kinematics of Rossby waves is also accompanied by nontrivial behavior of the mode in the vicinity of the focus (critical layer). If for the zonal case the focus is an absolute absorber of Rossby waves, and the transmission coefficient is exponentially small, then in the case of carrying out the *overshooting* mode of Rossby waves on a non-zonal flow, the problem becomes extremely sensitive to the initial data. In this case, absolutely opposite options are possible: from the complete absorption of the wave by the critical layer to its complete passage through the critical layer. In the second case, the transformation of the Rossby wave vertical mode does not occur.

In the previous work, the authors solved the problem of a strictly zonal flow [10]. An alternative, extremely simple, and physically understandable way to construct a reference solution was found. This approach was previously developed in plasma theory and later transferred by N. S. Erokhin and R. Z. Sagdeev to the water waves [20, 21]. Our solution is constructed using the Fourier analysis and some original changes of variables. In the appendix of [10], the constructed solution is identified and it is demonstrated that it exactly corresponds to the known solution in terms of a degenerate hypergeometric function of some complex argument with integration in the complex plane over a certain circle, as well as primary and secondary quantization using some reasoning about self-similarity. We show that behind all this conglomeration of plasma physics, obscure to specialists in the field of geophysics, there is a rather simple physical essence: there is a certain spatial curvature of the coordinates, and along with it, of the resulting solution.

In this regard, the critical layers are extremely stable, and this is the result of this work. We show that, at a qualitative level, the addition of non-zonality does not change the result obtained for the zonal case. This leads to the conclusion that the global processes of energy exchange in the ocean are concentrated in narrow regions, and resonance processes (where the background flow velocity is compared with the phase velocity of the wave disturbance) are extremely important for understanding the generation of waves and eddies by large-scale ocean currents (for more details, see [24]).

Thus, for the non-zonal problem of Rossby wave propagation in the vicinity of the focus, there is rather no analytical unambiguous prediction of the vertical mode behavior. The spectral problem is extremely sensitive to the initial data, and no limiting transition to the zonal case is observed.

Appendix
Vertical focusing in a short-wave approximation

Equation (3) for the flows with vertical variability can have solutions strongly localized in the vicinity of some fixed horizontal level $z = z_0$ where the phase velocity of the wave in longitudinal direction x coincides with the extremum of the main flow velocity field (the so-called critical layer) and attenuates in the rest of the region. The type of the velocity field extremum is deduced in the process of problem-solving and the boundary conditions for the Sturm – Liouville problem will be fulfilled automatically. In this case, the assumption that the localization level of the vertical mode does not coincide with the boundaries of the region vertically is fulfilled due to the exponential attenuation of the solution outside the localization level of the solution.

It is important to note that the solutions presented below do not claim to be unique and complete and describe only one of the possible scenarios determined by fulfilling the necessary conditions. The specificity of this problem is that the passage of the critical layer by the wave is usually associated with a spectrum transformation into the short-wave region. For Rossby waves on a zonal flow, the passage through the critical layer also unambiguously entails a short-wave transformation.

However, if the flow is not strictly zonal, two scenarios associated with passage through the critical layer are available. The first scenario is that the critical layer and the boundary of the transparency region coincide. When approaching the critical layer, a short-wave transformation of the wave occurs. Reaching the critical layer is asymptotic: the wave approaches the critical layer indefinitely in time. The second scenario is that the critical layer is located inside the region of transparency. Here the modes of passing the critical layer at finite values of wave numbers are possible and the wave passes the critical layer virtually unresponsive to its presence. Which of the two scenarios will be realized depends on the initial conditions of the problem. In this sense, the solution is highly sensitive to the selection of the initial data.

We consider the first scenario when a short-wavelength transformation occurs during the passage of the critical layer. For a solution strongly localized in the vicinity of the vertical level $z = z_0$ and attenuating in the rest of the region, equation (32) can be approximated as follows:

$$\left(\Psi_1\right)_{zz} + \left[P_0 + \frac{1}{2}P_2\left(z - z_0\right)^2\right]\Psi_1 = 0, \tag{A.1}$$

where

$$P_0 \equiv -S\left[k^2 + l^2 - \frac{K_\beta}{kU - \omega}\right]\Bigg|_{z=z_0}, \tag{A.2}$$

$$P_2 \equiv -S\left[\frac{K_\beta\, k\, U_{zz}}{\left(kU - \omega\right)^2}\right]\Bigg|_{z=z_0}, \tag{A.3}$$

$$K_\beta \equiv \left[k \left(\beta \cos\theta - \left(\frac{1}{S} U_z \right)_z \right) - l\, \beta \sin\theta \right] \Bigg|_{z=z_0} . \tag{A.4}$$

Equation (32) obtained earlier is analogous to equation (A.1), which has the following localized solutions:

$$\begin{gathered} \Psi_1 = A(k, \varepsilon y, \omega) \sum_{m=0}^{\infty} \left(2^n\, n! \sqrt{\pi} \right) (\Psi_1)_n , \\ (\Psi_1)_n = H_n \left[\left(-\frac{P_2}{2} \right)^{1/4} (z - z_0) \right] \exp\left[- \left(-\frac{P_2}{2} \right)^{1/2} \frac{(z - z_0)^2}{2} \right] . \end{gathered} \tag{A.5}$$

Here $H_n(x)$, $n = 0, 1, 2, ...$ are the Hermite polynomials.

From the relation (Kamke, E., 1961. *Handbook of Ordinary Differential Equations*. Moscow: Fizmatlit, 408 p.)

$$(2n+1) = P_0 \left(-\frac{P_2}{2} \right)^{-1/2} \tag{A.6}$$

we find the dispersion relations

$$\omega = \frac{-K_\beta + (2n+1) S^{-1/2} \left[K_\beta\, k\, U_{zz} \big|_{z=z_0} \right]^{1/2}}{k^2 + l^2} + k\, U \quad \text{at} \quad (\omega - k\, U) < 0, \tag{A.7}$$

$$\omega = \frac{-K_\beta - (2n+1) S^{-1/2} \left[K_\beta\, k\, U_{zz} \big|_{z=z_0} \right]^{1/2}}{k^2 + l^2} + k\, U \quad \text{at} \quad (\omega - k\, U) > 0. \tag{A.8}$$

Expressions (A.7) and (A.8) are analogous to the dispersion relation for the barotropic problem. Let us note that the following restriction was made in this case:

$$P_0 > 0 . \tag{A.9}$$

Taking expression (A.2) into account, we find that the signs of the expressions K_β and $(\omega - k\, U)$ coincide. Next, the following condition must be satisfied

$$P_2 < 0\,. \tag{A.10}$$

Taking into account (A.3), we obtain $K_\beta\, k\, U_{zz}\big|_{z=z_0} < 0$. Omitting the details, we can state the following result. Dispersion relation (A.7) corresponds to the case where the wave passes through the critical layer first, then bounces off a region of large background flow velocities, and then approaches the critical layer from the back side, asymptotically approaching the main flow velocity maximum on the large side. Dispersion relation (A.8) corresponds to the case where the wave approaches the critical layer asymptotically approaching the main flow velocity minimum on the lower side. At the zonal flows $(\theta = 0)$, the critical layer is realized only for westward flows and only in the form of asymptotics (A.8). (Here we adhere to the terminology adopted in oceanography, where a westward flow is a flow directed to the west, whereas in atmospheric physics a westward wind is a wind propagating from the west). For westward zonal flows, the vertical level z_0 on the z coordinate, at which the mode is focused, is the level of the absolute maximum velocity of the main flow. In this case, reaching the critical layer is asymptotic in time and is accompanied by a short-wave transformation.

For non-zonal flows, the situation breaks down into particular cases depending on the initial parameters of the problem. In this case, critical layers can be realized both for positive values of the main flow velocity and for negative ones. What exactly will happen to the wave in the vicinity of the critical layer for the non-zonal case depends on the initial position of the wave (from which side and at what angle the wave approaches the critical layer) and on the parameters of the main flow velocity field in the vicinity of the extremum along the vertical coordinate. On a non-zonal flow, the following options are possible: the passage of the wave through the critical layer and vertical focusing on the extremum of the main flow velocity field.

We shall note one important result, which is obtained from formula (A.7). For both zonal and non-zonal flows, the vertical variability of the velocity field leads to one common property: if the vertical structure of the mode is focused on the extremum of the velocity field, then there is a limit on the number of modes that can be inscribed in the vicinity of this level:

$$(2n+1) < \frac{|K_\beta|}{S^{-1/2}\left[K_\beta\, k\, U_{zz}\big|_{z=z_0}\right]^{1/2}}\,. \tag{A.11}$$

This fact is well known for Rossby waves (LeBlond, P.H. and Mysak, L.A., 1978. *Waves in the Ocean*. Amsterdam: Elsevier, 602 p.). However, in this problem, the baroclinicity can lead to the absence of the vertical mode at all. Such cases in the numerical calculation of the vertical eigenvalue problem were considered in the works by P.D. Killworth and J.R. Blundell (Killworth, P.D. and Blundell, J.R.,

2003. Long Extratropical Planetary Wave Propagation in the Presence of Slowly Varying Mean Flow and Bottom Topography. Part I: The Local Problem. *Journal of Physical Oceanography,* 33(4), pp. 784-801. doi:10.1175/1520-0485(2003)33<784:LEPWPI>2.0.CO;2; Killworth, P.D. and Blundell, J.R., 2005. The Dispersion Relation for Planetary Waves in the Presence of Mean Flow and Topography. Part II: Two-Dimensional Examples and Global Results. *Journal of Physical Oceanography,* 35, pp. 2110-2133. doi:10.1175/JPO2817.1). For solutions (A.5), we introduce the definition of the vertical variability scale of solution D:

$$D^{-1} \equiv \left(-\frac{P_2}{2}\right)^{1/4} = \left[\frac{S\,K_\beta\;k\,U_{zz}\big|_{z=z_0}\left(k^2+l^2\right)}{\left(-K_\beta \pm (2n+1)S^{-1/2}\left[K_\beta\;k\,U_{zz}\big|_{z=z_0}\right]^{1/2}\right)^2}\right]^{1/4}. \qquad \text{(A.12)}$$

Formula (A.12) gives a characteristic scale of vertical variability and also determines the so-called self-similarity of the solution. It is easy to see that in the short-wavelength limit the vertical D and longitudinal L scales of the solution are linked by the relation $D^{-1} \sim L^{1/2}$.

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