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Critical Levels of the Sea Breeze Circulation within the Framework of Linear Theory

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Abstract
Purpose. The present paper is aimed at solving (within the framework of linear theory) a problem on the influence of critical levels on an inertia-gravity wave with the scale typical of the sea breeze circulation in the presence of an average background synoptic flow (perpendicular to the coast) with a vertical shear.
Methods and Results. To solve the problem, the generalized Taylor – Goldstein equation was used with the regard for rotation. The coefficients' behavior, having been analyzed, showed that at the equator, for a gravity internal wave (generated by a heat source on the surface) with the daily frequency, there existed a single critical level, at which the wave was absorbed. In the tropics, an inertia-gravity wave of the daily period passes through two critical levels and the attenuation region located between them. At the mid-latitudes, the generated wave, starting from the surface, is in the attenuation zone, then passes the critical level and propagates above it. To analyze the solution behavior, the Cauchy problem on the wave passage through the critical level was solved. The solution near the critical level was obtained numerically.
Conclusions. For the selected values of stratification and background wind speed, the absorption coefficients of the vertical momentum flux were calculated for the equator and those of the vertical angular momentum flux – for the tropics and middle latitudes. The absorption coefficient value at the equator is in complete agreement with the results obtained in the earlier published papers. Comparing the values of the absorption coefficients of the vertical momentum flux/vertical angular momentum flux at different latitudes, one can note that the strongest attenuation takes place at 15°, and the weakest one – at 45°.

Keywords: linear theory, sea breeze circulation, internal inertia-gravity waves, critical level, stratification, shear flow

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Introduction

Breeze circulation is a common phenomenon near the shores of water bodies. Formed in coastal regions, the breeze affects the weather and climate of these areas.

Various approaches are used to study breeze circulation: analytical theory [1–3], field measurements [4–6], laboratory experiment [7, 8], numerical simulation [9–11], and similarity theory [12–14].

The main features and mechanisms of breeze circulation are well described by the linear theory, in which the breeze is an internal inertia-gravity wave generated by a heat source in the atmospheric boundary layer as a result of daytime heating and nighttime cooling. The linear theory of breeze circulation has been developed since the late 1940s [1, 15]. We note several important works in which two different approaches to the description of the heat source were used.

In the first paper [16], the classical statement of the linear problem of breeze circulation was applied: the breeze developed after the formation of the atmospheric surface layer as a result of convective turbulent mixing. Turbulent mixing was described using constant coefficients of turbulent viscosity and thermal conductivity. The resulting equation for the stream function was of the sixth order, which greatly complicated the analysis of solutions. The linear equations themselves had to be solved numerically.

In the second paper [2], the vertical and horizontal heat distribution was specified explicitly and only the dynamic response to the heat source was studied. An important result of this work was the identification of two regimes of breeze circulation depending on latitude – two beams of internal inertia-gravity waves emitted from the coast in the tropics and a vertically localized breeze cell in middle latitudes.

The simplicity of the approach of [2] made it possible to consider in a similar way the interaction of the breeze circulation with a vertically uniform synoptic wind. This problem was considered in several works without taking into account the Coriolis force [17] and in a more general case [18, 19]. The presence of an average background synoptic wind leads to the following two effects: firstly, the waves radiated from the coastline become asymmetric; secondly, a train of standing short (quasi-stationary) waves is generated above the coast line, similar to the waves observed when wind flows around a mountainous terrain.

In real synoptic situations, the wind profile is rarely vertically uniform. As a rule, there is a vertical wind velocity shear. It is known that the presence of this shear leads to the formation of critical levels at certain heights, where the frequency of internal waves, with regard to the Doppler shift, vanishes [20].

The problem of the internal waves' behavior in the vicinity of the critical level is classical and has been studied since the 1960s [20–24]. Mathematically, the problem is reduced to solving the vorticity equation for small perturbations of a stratified shear flow (the Taylor – Goldstein equations). Within the framework of this equation, the problem of the development of instability for small values of the Richardson number and the problem of the behavior of small stable perturbations were considered. It should be noted that in this work, in contrast to [20–24], the inhomogeneous Taylor – Goldstein equation with the heat source gradient on the right side is used.

The behavior of purely internal gravity waves in the vicinity of the critical level in the absence of the Earth's rotation is described in [20]. One of the main results of this work is the description of the intense absorption of the internal wave at the critical level, which increases with the Richardson number.

Accounting for the rotation of the Earth complicates the problem. A generalization of the Taylor–Goldstein equation, taking into account the Earth's rotation, was performed in [22, 23]: the splitting of the critical level into two was found, while the wave frequency, taking into account the Doppler shift at the critical level, is equal to plus/minus the Coriolis parameter. The amplitude of the wave decreases when passing through the critical level in the same way as without taking rotation into account. In this case, the influence of rotation is significant only near the critical level, and far from it, the solutions with and without rotation coincide.

In the breeze circulation problem, it is necessary to take into account the Earth's rotation, since the diurnal frequency of the heat source of this circulation is comparable to the value of the Coriolis parameter.

The problem of the critical level effect on the breeze circulation at the equator was considered relatively recently [25]. The main result was that at certain heights, critical layers, that absorb internal waves and limit the height of the breeze circulation beam propagating downstream, are formed.

The purpose of this work is to solve in a linear formulation the problem of the critical levels' effect on an inertia-gravity wave with a scale characteristic of breeze circulation in the presence of a background synoptic wind with a vertical shear directed perpendicular to the coastline. This work is an extended continuation of the study presented in two conference papers [1].

This work is aimed at generalizing the results obtained earlier for the equator [25] to the general case of an arbitrary latitude: we study the passage through critical levels and absorption of a breeze inertia-gravity wave in the tropics and middle latitudes and determine the differences between them.

The paper presents a short derivation of the Taylor – Goldstein equation generalization taking into account the Earth's rotation, describes a numerical method for solving this equation and the main types of critical levels that occur in the tropics and mid-latitudes, obtains and discusses solutions for different latitudes at typical values of stratification and wind velocity shear in atmosphere.

Generalization of the Taylor – Goldstein equation with regard to the Earth's rotation

The behavior of small perturbations of a stratified shear flow is described by the linearized Taylor – Goldstein vorticity equation. At Richardson numbers Ri < 1/4, this equation describes the linear stage of instability development; at Ri > 1/4 – the propagation of internal waves in a stratified shear flow. A generalization to the case of taking into account the Earth's rotation was made in [22, 23].

For the convenience of presenting the results of this work, below is a brief derivation of the generalization of the Taylor – Goldstein equation, taking into account the Earth's rotation.

We shall consider the behavior of small perturbations of a stratified plane-parallel horizontal flow with a buoyancy frequency $N(z)$ with a vertical shear with

[1] Shokurov, M.V. and Kraevskaya, N.Yu., 2020. [Linear Theory of Breeze Circulation for an Arbitrary Background Wind Velocity Profile and Stratification]. In: MHI, 2020. *Seas of Russia: Studies of Coastal and Shelf Zones (XXVIII Coastal Conference).* Sevastopol: MHI, pp. 209-210 (in Russian); Shokurov, M.V. and Kraevskaya, N.Yu., 2022. Critical Levels of the Breeze Inertia-Gravity Wave. In: MHI, 2022. *The Seas of Russia: Challenges of the National Science*. Sevastopol: MHI, pp. 166-168 (in Russian).

a velocity profile $U(z)$ in the Boussinesq and f-plane approximations without taking into account viscosity and thermal conductivity. We shall consider a two-dimensional problem in the plane (x, z). The background flow $U(z)$ is directed along the x axis, i.e., $V(z) = 0$. Perturbations propagate only in the direction of the x axis, i.e., $\partial/\partial y = 0$, $k_y = 0$. More general cases are considered in the works [22, 23].

The equations of motion, heat transfer and continuity are as follows:

$$\frac{\partial u}{\partial t} + U(z)\frac{\partial u}{\partial x} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t} + U(z)\frac{\partial v}{\partial x} + fu = 0,$$

$$\frac{\partial w}{\partial t} + U(z)\frac{\partial w}{\partial x} - b = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}, \qquad (1)$$

$$\frac{\partial b}{\partial t} + U(z)\frac{\partial b}{\partial x} - vf\frac{\partial U(z)}{\partial z} + N^2(z)w = Q(x,z,t),$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

where u, v, w are the velocity vector components; $U(z)$ is the background wind velocity; f is the Coriolis parameter; ρ_0 is the density in the ground state; p is pressure; b is buoyancy; N is the buoyancy frequency; Q (x, z, t) is a function describing the heat source distribution.

In the equation for buoyancy in system (1), the third term describes the alongshore heat (buoyancy) advection; in this term, the background alongshore buoyancy gradient is expressed in terms of the vertical shear of the background synoptic wind perpendicular to the coastline in accordance with the thermal wind equation.

As mentioned earlier, to set a small perturbation, a heat source Q (x, z, t) is used, which characterizes its change in the atmospheric boundary layer during the day. The heat source can have any spatial distribution horizontally and vertically, and the time dependence is defined as $\sin(\omega t)$.

Based on the continuity equation, we introduce the stream function: $u = \partial\psi/\partial z$, $w = -\partial\psi/\partial x$. The coefficients of the system of equations (1) do not depend on the horizontal coordinate x and time t, so we will look for a solution in the form $\psi \sim \exp\big(i(\omega t - kx)\big)$.

Although the heat source is spatially inhomogeneous, it is possible to perform its Fourier transformation on x and further consider each harmonic k separately. The frequency ω in the problem of breeze circulation is diurnal: $\omega = 2\pi/T$, where T is a solar day.

Eliminating successively all the variables, except for the stream function, we obtain the vorticity equation

$$(f^2 - \omega'^2)\psi_{zz} + \frac{2kf^2U_z}{\omega'}\psi_z + \big((\omega'^2 - N^2)k^2 - \omega' kU_{zz}\big)\psi = ikQ, \qquad (2)$$

where $\omega' = \omega - Uk$ is an eigenfrequency of the wave in a fixed frame of reference. At $f = 0$, this equation reduces to the Taylor – Goldstein equation; at $U = 0$, it describes the propagation of internal waves in a stationary stratified medium, purely gravity wave at $f = 0$ or inertia-gravity wave at $f \neq 0$. In a homogeneous fluid ($N = 0$) this equation reduces to the Rayleigh equation describing stability (instability) of shear flows.

We denote the coefficients of equation (2):

$$A(z) = f^2 - \omega'^2, \tag{3}$$

$$B(z) = \frac{2kf^2U_z}{\omega'}, \tag{4}$$

$$C(z) = (\omega'^2 - N^2)k^2 - \omega' k U_{zz}. \tag{5}$$

Then the equation (2) takes the following form:

$$A\psi_{zz} + B\psi_z + C\psi = ikQ. \tag{6}$$

This is a non-homogeneous second-order differential equation with variable coefficients. The behavior of the solution $\psi(z)$ depends on the behavior of the coefficients $A(z)$, $B(z)$, $C(z)$.

Depending on the signs of $A(z)$ and $C(z)$, different types of solutions will be obtained: if $A(z)$ and $C(z)$ have the same signs, then the solution will correspond to a propagating wave, if the signs are different, then the solution will describe attenuation. The sign change of $C(z)$ coefficient at the point z_r corresponds to the reflection of internal waves at this level.

The vanishing of the coefficient $A(z)$ at the point z_c gives a more complicated situation – the emergence of a critical level at which the vertical wave number k_z tends to infinity. The behavior of the solution near the critical level (in the critical layer) has been studied in detail in many papers. In particular, an analytical solution was obtained in a small neighborhood of the critical level, where the velocity profile can be considered linear (for a stationary wave, $\omega = 0$ and without taking into account the Earth's rotation, $f = 0$ in [21]). To obtain an analytical solution, as a rule, the Frobenius method is used, while the solution can be obtained in the vicinity of the critical level, the singular part of which has the form

$$\psi = (z - z_c)^{1/2 \pm i\mu}, \qquad \mu = (\mathrm{Ri} - 1/4)^{1/2}, \tag{7}$$

where $\mathrm{Ri} = N^2/U_z^2$ is the Richardson number.

In the stable case Ri > 1/4, μ is a real value, formula (7) defines two linearly independent solutions of equation (6), from which it is necessary to make linear combinations for $z > z_c$ and $z < z_c$.

The critical levels of equation (2) and the behavior of solutions at $f \neq 0$ were studied in [22, 23].

Method for the numerical solution of the Taylor – Goldstein equation taking into account the Earth's rotation

To analyze the behavior of the solution near the critical level z_c, we solved the Cauchy problem on the passage of a wave through the critical layer. To do this, at an arbitrary point in height, the values ψ and $\varphi = \psi_z$ were set.

In the presence of critical levels $A(z_c) = 0$, the solution at the point z_c becomes singular: $k_z = \infty$, the wavelength λ_z tends to zero in the vicinity of z_c, which creates difficulties for a numerical solution limited by the grid spacing. One way to overcome this difficulty is to introduce artificial viscosity: adding a small imaginary part $\omega \rightarrow \omega + i\omega_i$ to the frequency ω [20, 22]. In this case, the solution in the critical layer will have a finite wavelength $\lambda_z \neq 0$. Choosing a sufficiently small discretization step $\Delta z << \lambda_z$, one can obtain the correct numerical solution. Further, the solutions obtained by the Cauchy method will be analyzed both for the classical critical layer ($f = 0$) and for the critical layer taking into account the Earth's rotation ($f \neq 0$).

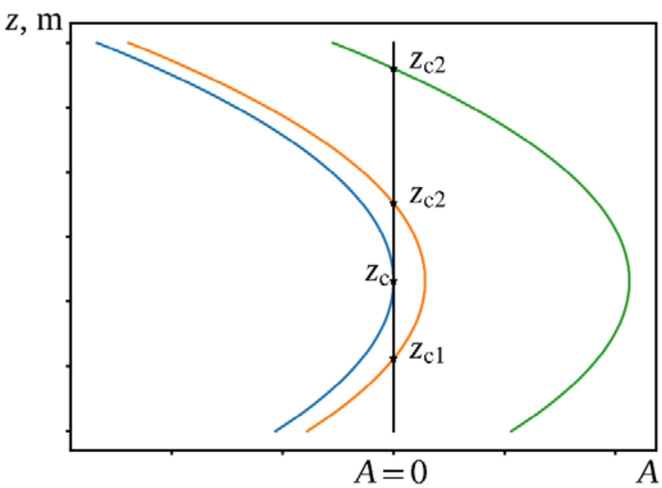
Analysis of the behavior of the Taylor – Goldstein equation coefficients taking into account the Earth's rotation and the location of critical levels

Before obtaining numerical solutions, we consider the dependence of the coefficients $A(z)$, $B(z)$, $C(z)$ on z for typical atmospheric profiles $N(z)$ and $U(z)$ for a certain horizontal harmonic with wavenumber k. Further, we assume that the stratification is constant ($N(z)$ = const), and the velocity profile is linear ($U(z) = U_z z$).

For a linear background wind profile, the coefficient $A(z)$ is defined as

$$A(z) = f^2 - \omega'^2 = f^2 - (\omega - kU_z z)^2. \quad (8)$$

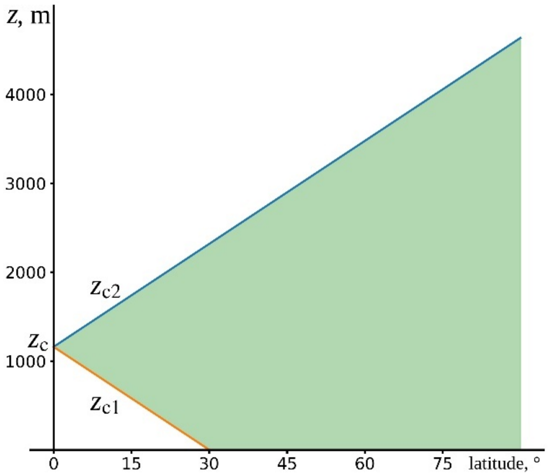
As can be seen from (8), the dependence of the coefficient $A(z)$ on z is quadratic. At $f = 0$ (at the equator) there is one critical level $z_c = \omega/(kU_Z)$. The behavior of $A(z)$ and the location of the critical level z_c are given in Fig. 1.



F i g. 1. Position of critical levels for the equator (blue curve), tropics (orange curve) and mid-latitudes (green curve)

In the tropics (latitude φ < 30°) there are two critical levels: $z_{c1,2} = (\omega \pm f)/(kU_Z)$. Since $f < \omega$, both critical levels are located above the Earth's surface. At φ = 30°, the lower critical level z_{c1} touches the Earth's surface, and at middle latitudes φ > 30°, therefore, z_{c1} is under the Earth's surface and does not affect the solution of equation (6).

In Fig. 2, a summary picture of the location of critical levels depending on the latitude from the equator to the pole, is demonstrated.



F i g. 2. Position of critical levels depending on a latitude

We consider the behavior of the coefficient $B(z)$. In accordance with (4), this coefficient turns to infinity at the level $z_3 = (\omega - 2fkU_Z)/(kU_Z)$, located in the middle between the levels z_{c1}, z_{c2}. However, the solution at this level is not singular [22, 23]; therefore, it is not considered as critical. This level is formed at all latitudes, at the equator it coincides with the critical level, its location does not depend on latitude, since the second term in the numerator takes values that are much lower than the values of the diurnal frequency.

We proceed to consider the dependence of the coefficient $C(z)$ on the height. With a linear profile, which is considered in this paper, $U_{zz} = 0$, therefore (5) takes the form $C(z) = (\omega'^2 - N^2)k^2$. Coefficient $C(z)$ changes sign at two points: $z_{r1,2} = (\omega \pm N)/(kU_Z)$.

The Earth's atmosphere is characterized by $N >> \omega$, these frequencies differ by about two orders of magnitude, so one of these reflection levels is located under the Earth's surface, and the second is at a high altitude, several hundred kilometers, and does not affect the breeze circulation. The location of the reflection levels z_{r1}, z_{r2} does not depend on latitude.

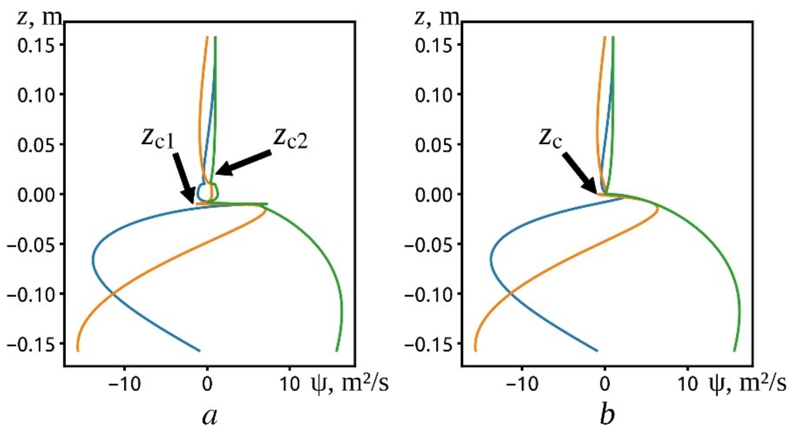
Thus, the coefficient $C(z)$ is always negative, so the square of the vertical wave number $k_z^2 = C\,(z)/A\,(z)$ is positive where $A(z)$ is negative (wave propagation), and negative where $A(z)$ is positive (wave attenuation). On the summary diagram (Fig. 2), the region of wave attenuation is marked with color.

Analyzing Fig. 2, we see that at the equator the internal gravity wave generated by the heat source on the surface with a diurnal frequency reaches the critical level z_c, at which it is absorbed. In the tropics, an inertia-gravity wave of a diurnal period radiated from the surface passes through two critical levels z_{c1}, z_{c2} and the attenuation region located between them. At mid-latitudes, the inertia-gravity wave generated on the surface is in the attenuation zone $0 < z < z_{c2}$, passes the critical level z_{c2}, and then propagates in the zone $z > z_{c2}$.

We note that, according to equation (2), as the critical level is approached, the vertical wavenumber increases. Depending on the type of solution, wave or exponential, an increase in this number when approaching the critical level will give different behavior of the solution: with a wave solution, a decrease in the vertical wavelength will be observed, with an exponential type of solution, the exponent will change abruptly, and therefore the solution will have a sharp transition from an exponent with a lower decay rate to an exponent with a higher decay rate.

Results

Before performing calculations for the breeze circulation, the numerical finite difference scheme was tested by comparing the results with the results of the numerical solution from [22] at $f = 0$ and $f \neq 0$ (Ri = 1). In Fig. 3, the profiles of the stream function, the behavior of which is similar to the behavior of the vertical velocity component in Fig. 3 in [22], are demonstrated.



F i g. 3. Dependence of the real (blue curve) and imaginary (orange curve) parts, and modulus (green curve) of the stream function on height at $f = 10^{-2}$ (*a*) and $f = 0$ (*b*) (Ri = 1)

It can be seen that the values near the surface exceed those obtained in [22], this behavior is associated with the effect of the reflected wave at Ri = 1. However, numerical experiments have shown that the effect of the reflected wave for values typical of the real atmosphere at Ri = 100 is insignificant, therefore, the chosen numerical method is suitable for describing the wave near the critical level. The Cauchy problem method is going to be used further in the study of critical levels for breeze circulation.

On the basis of preliminary test calculations, the following values of the grid spacing and the imaginary addition to the frequency were chosen: $\Delta z = 10^{-3}$ m, $\omega_i = 2 \cdot 10^{-10}$ 1/s. The choice of ω_i is due to the possibility of correctly describing the shortest wavelengths at the chosen grid spacing.

For further experiments, the buoyancy frequency was chosen to be $N = 10^{-2}$ 1/s; this is a typical value for the Earth's atmosphere. The horizontal wave number is chosen to be $2\pi/100$ 1/km, which corresponds to the characteristic horizontal scale of the breeze circulation cell in the Earth's atmosphere [2].

To set the initial disturbance, the information on the typical value of the horizontal velocity component in the breeze circulation, which is 0.01 m/s at an amplitude of the volume heat source of 1 W/m^2 was applied. The typical value of the horizontal velocity component was estimated using the linear theory at $N = 10^{-2}$ 1/s, $U = 0$ m/s.

Three numerical experiments were performed at three different latitudes: 0; 15 and 45° at background wind shear $U_z = 10^{-3}$ 1/s. The heights of critical levels and the absorption intensity during the passage of critical levels are given in the table.

According to [20, 22], when the critical level is passed, the wave is absorbed; the theoretical absorption coefficient is determined by the Richardson number: $\lambda_{theor} = \exp(2\pi\mu)$. As was found in [22], rotation has almost no effect on the absorption value when passing through the critical level, which is expressed in a small difference in the wave attenuation coefficients with and without taking into account the Earth's rotation.

Position of the critical levels and absorption intensity at their passage

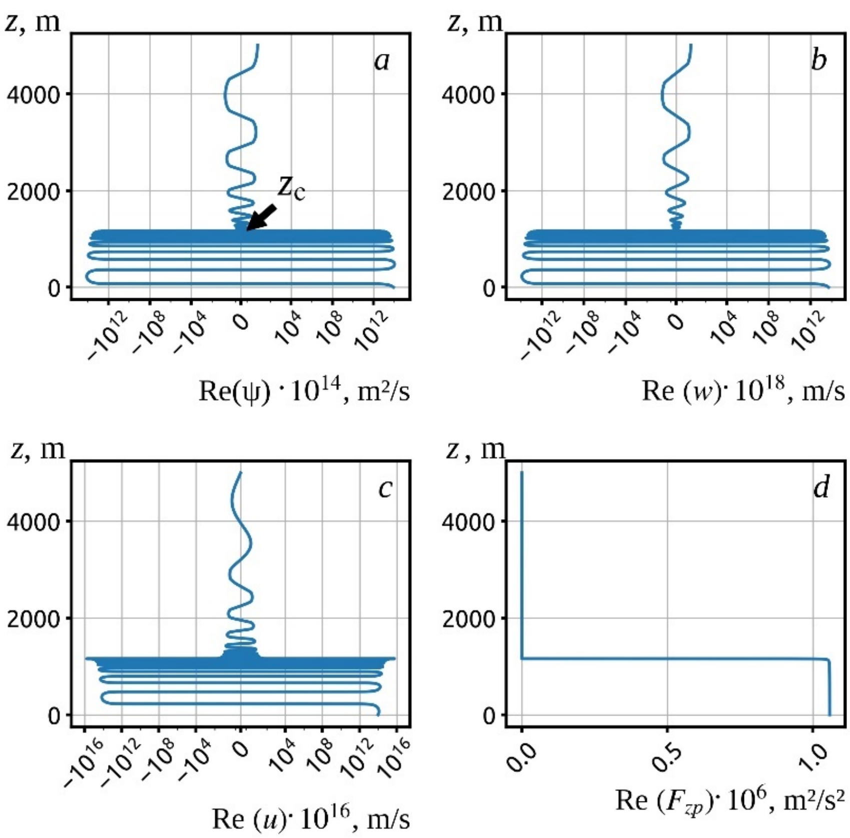
Parameters	φ°		
	0	15	45
z_{c1}	1160.24	559.66	–480.59
z_{c2}	1160.24	1760.82	2801.07
Ri	100.00	100.00	100.00
μ	9.99	9.99	9.99
λ_{theor}	$1.79 \cdot 10^{27}$	–	–
λ_{exp}	$1.79 \cdot 10^{27}$	$4.46 \cdot 10^{27}$	$2.32 \cdot 10^{14}$

Critical levels and intensity of absorption while passing them

We consider the behavior of a breeze inertia-gravity wave propagating along the flow at the equator. In this experiment, the entire thickness of the atmosphere is a wave propagation zone, while there is one critical level, its height is $z_c = 1160.24$ m.

In Fig. 4 the vertical profiles of the variables $\psi(z)$, $u(z)$, $w(z)$ are given. The alongshore velocity component $v(z)$ at the equator is equal to zero. The stream function changes, as does the vertical velocity (Fig. 4, *a*, *b*), these variables are in antiphase and differ by the value of the horizontal wave number $w = ik\psi$. On both sides of the critical level, a wave type of solution is observed. As the wave approaches the critical level, its amplitude and vertical wavelength decrease, and after its passage they increase again.

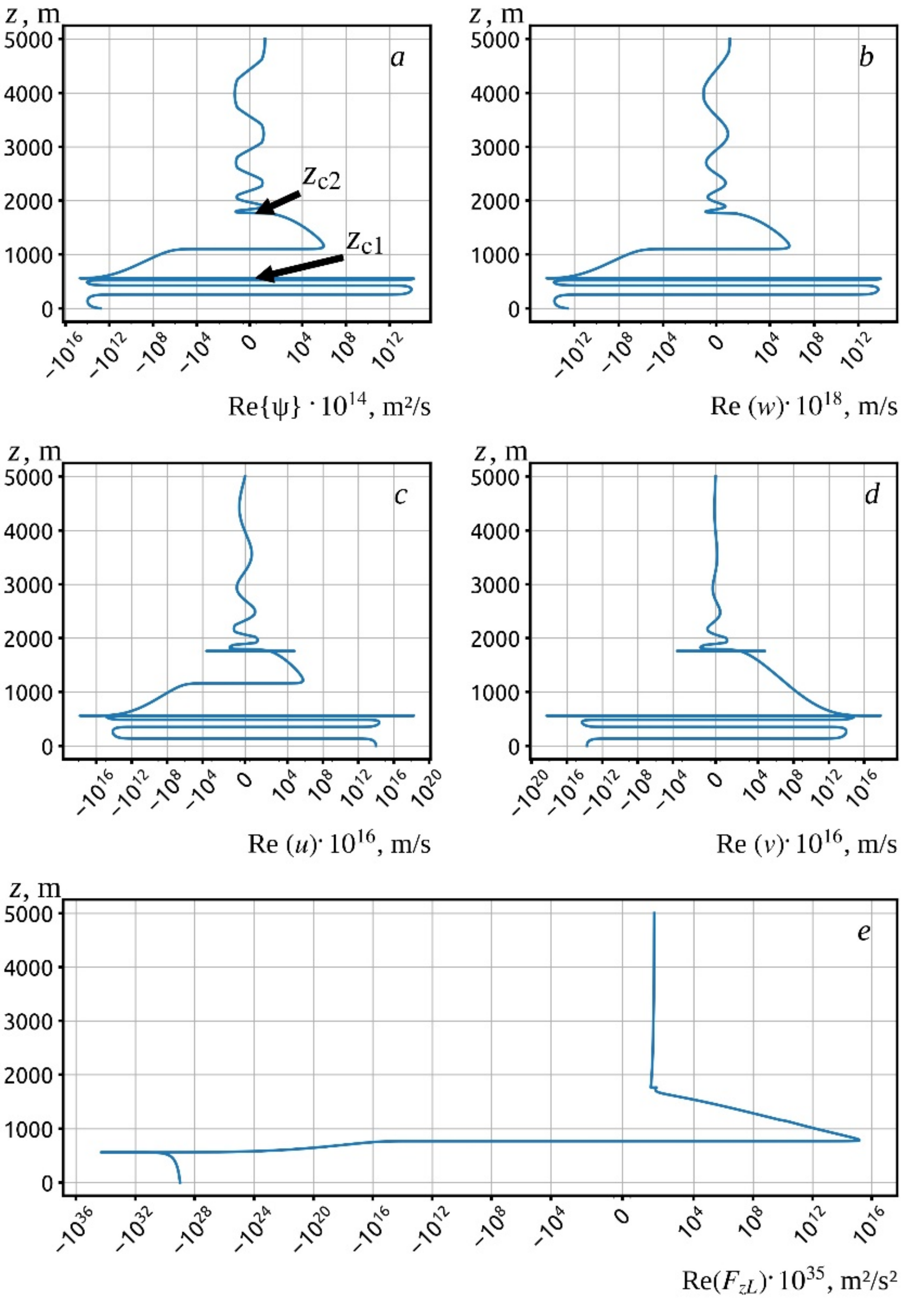
An oscillatory motion is also observed in the horizontal velocity component (Fig. 4, *c*); when approaching the critical level from both sides, the wave amplitude increases. After passing the critical level, the amplitude sharply decreases by several orders of magnitude.



F i g. 4. Dependence real parts of the stream function (*a*), two velocity components (*b*, *c*) and vertical momentum flux (*d*) on the height at the equator

The vertical momentum flux $F_{zp}(z)$ per unit mass was determined as follows: $F_{zp}(z)$ = Re(w*u), where Re is the real part of the quantity; * is complex conjugation. The vertical momentum flux (Fig. 4, *d*) takes a constant value above and below the critical level, and intense wave absorption is observed at the critical level. The experimental attenuation coefficient was calculated using the ratio of the values of the vertical momentum flux at a height of 30 and 4750 m and was calculated as $\lambda_{exp} = 1.79 \cdot 10^{27}$, which is in full agreement with the theoretical value (Table).

In the tropics, a wave moving from the surface will pass through two propagation layers ($0 < z < z_{c1}$ and $z > z_{c2}$) and one attenuation layer ($z_{c1} < z < z_{c2}$) between them. In this experiment, there are two critical levels, their heights are given in the Table.



F i g. 5. Dependence real parts of the stream function (*a*), three velocity components (*b* – *d*) and vertical angular momentum flux (*e*) on the height at 15° latitude

In the profile of the stream function (Fig. 5, *a*) and the vertical velocity component (Fig. 5, *b*), oscillations are observed near the surface; when approaching the critical level, their amplitude increases, and the vertical wavelength decreases. Due to the existence of an exponential decay zone in the wave propagation region above the second critical level, the amplitude and vertical wavelength have a limited value, and do not tend to zero as at the equator.

In the horizontal velocity component (Fig. 5, *c*), near both critical levels, the amplitude increases, but a limited number of oscillations is observed. The behavior of $u(z)$ in the attenuation region is similar to that observed as a stream function and the vertical velocity component.

An interesting feature in the profiles $\psi(z)$, $w(z)$, $u(z)$ is the presence of a sign change level in the attenuation region.

In the horizontal velocity component $v(z)$ (Fig. 5, *d*), the amplitude increases near both levels. When approaching the first critical level from below and moving away from the second critical level, when the wave propagates upward, an increase in the vertical wavelength is observed. In the damping zone, $v(z)$ does not change sign, unlike other variables.

As noted in [22], in the frame of reference, taking into account rotation, instead of conserving the vertical flux of momentum, the vertical flux of angular momentum would be conserved, which was calculated per unit mass according to formula (18) from the mentioned work: $F_{zL}(z) = \mathrm{Re}(w^*(v + 2\Omega\ \xi) - \mathrm{Re}(w^*(u - - 2\Omega\eta))$, where $F_{zL}(z)$ is the vertical angular momentum flux, ξ and η are the particle displacements from the initial position in x and y, respectively. It is assumed that the particle is displaced relative to the origin. Since a wave with a certain (diurnal) frequency is considered, the particle displacement is calculated as $\xi = i\ \omega$, $\eta = i\ \omega$.

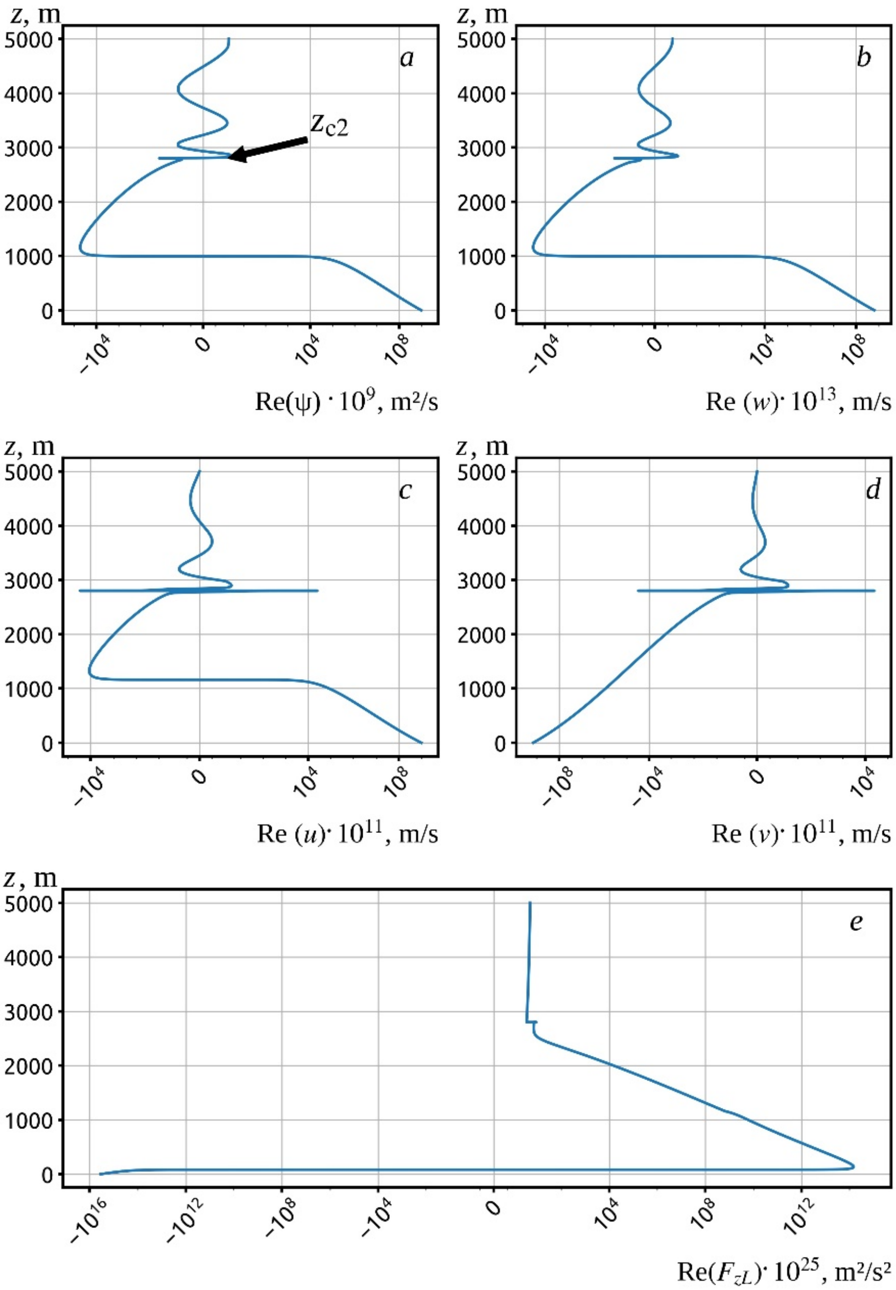
The vertical angular momentum flux (Fig. 5, *e*) takes a constant negative value below the first critical level and a positive constant value above the second critical level in the regions corresponding to wave propagation.

To calculate the experimental attenuation coefficient, the ratio of the vertical angular momentum flux values in areas where it takes on an approximately constant value at a height of 30 and 4750 m was applied: $\lambda_{\text{exp}} = 4.46 \cdot 10^{27}$, which exceeds the value of the coefficient at the equator by 2.49 times (Table). In the tropics, wave absorption is more intense than at the equator.

Since there is no analytical theory for the case when $f \neq 0$, we will discuss the result obtained at the empirical level. The structure of critical levels is similar to that obtained in [22] at $f = 10^{-2}$ (Fig. 3), but there is a difference consisting in the appearance of a sign change level between two critical levels in all considered variables, except for the velocity component $v(z)$.

At middle latitudes, the inertia-gravity wave generated on the surface first exists in the attenuation zone $0 < z < z_{c2}$, then it propagates in the zone $z > z_{c2}$. In this experiment, the first critical level $z_{c1} = -480.59$ m is underground, which means that the wave will pass through only one critical level $z_{c2} = 2801.07$ m.

In the profiles of the stream function and the vertical velocity component (Fig. 6, *a*, *b*), as well as at 15° in the transition from the exponential type of solution to the wave one, the change in the wave amplitude is not so sharp: the vertical wavelength has a finite value, and does not tend to zero, as at the equator. In the horizontal velocity component (Fig. 6, *c*), an increase in the amplitude near the critical level is observed.



F i g. 6. Dependence real parts of the stream function (*a*), three velocity components (*b* – *d*) and vertical angular momentum flux (*e*) on the height at 45° latitude

Let us note that in all three variables $\psi(z)$, $u(z)$, $w(z)$ a level of sign change in the exponential region of the solution is observed.

In the profile of the horizontal velocity component $v(z)$ (Fig. 6, *d*) near the critical level, just as in the velocity component $u(z)$, an increase in amplitude is observed. The component $v(z)$ differs from other variables in that there is no level of sign change in the zone of exponential decay.

The vertical angular momentum flux (Fig. 6, *e*) takes a constant value in the region of wave propagation above the second critical level.

In the wave attenuation region, the angular momentum flux is negative near the surface, then it changes sign and decreases, remaining positive even after passing the critical level.

At mid-latitudes, there is only one region where the vertical angular momentum flux is constant, but for comparison with other experiments, the experimental absorption coefficient was calculated. For the calculation, as well as for other experiments, the ratio of the vertical angular momentum flux values near the surface and near the upper boundary, where its value was constant, was applied. The calculated experimental absorption coefficient equals $2.32 \cdot 10^{14}$, which is $0.77 \cdot 10^{13}$ times less than the theoretical coefficient at the equator (Table).

Comparing the values of the absorption coefficients at different latitudes, it can be noted that the greatest attenuation occurs at 15°, and the smallest – at 45°. At mid-latitudes, there will be less attenuation from an evanescent wave propagating from the surface and passing through a single critical level than from a propagating wave passing through a single critical level at the equator, or a wave passing through two critical levels in the tropics.

At the equator and in the tropics in an atmosphere without a background wind, a breeze inertia-gravity wave propagates vertically to infinity within the framework of the linear theory. As shown in the present study and in [25], the presence of a critical level reduces the height of the breeze circulation beam propagating downstream.

Since in the middle latitudes the breeze inertia-gravity wave decays with height and in a stationary atmosphere, the critical level effect will be observed when its height is less than the breeze circulation height, which is determined by the height of the atmospheric boundary layer.

Conclusion

In this paper, within the framework of the linear theory, we consider the problem of critical levels' effect on the inertia-gravity wave generated on the surface by a heat source with a scale characteristic of breeze circulation. Critical levels for such a wave were formed due to the existence in the atmosphere of an average background synoptic wind with a vertical shear directed perpendicular to the coastline. To solve the problem, the Taylor – Goldstein equation was applied, taking into account the Earth's rotation, which was solved numerically by the Cauchy problem method.

As a result of the analysis of the equation coefficients, it was found that a single critical level is observed at the equator. The gravitational internal wave with a diurnal frequency generated by a heat source on the surface reaches this critical level, where it is absorbed. A comparison of the theoretical and experimental absorption coefficients of the vertical momentum flux showed complete agreement.

In the tropics, there are two critical levels between which an attenuation region is located. An analysis of the vertical angular momentum flux revealed that the inertia-gravity wave of the diurnal period emitted from the surface is attenuated more strongly than at the equator.

There is only one level for the wave more than 30° polar, since the second level is below the Earth's surface. At mid-latitudes, an inertia-gravity wave generated on the surface first exists in the attenuation region, after which it propagates above a single critical level. At these latitudes, the ratio of the vertical angular momentum flux value near the Earth's surface and the value in the region where the angular momentum flux is constant takes a minimum value for all three latitudes.

Thus, the results of this work are consistent with the works of previous years and contribute to the development of topics on the effect of background wind shear on inertia-gravity waves and breeze circulation.

For a more realistic description of the breeze circulation within the framework of the linear theory, it is necessary to take into account that the background wind is rarely strictly perpendicular to the coast, it usually forms an angle with the coast. It should also be taken into account that the background wind and stratification can greatly vary vertically and in time.

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