$$\overline{\phi}^x_{i,j,k} = \frac{\phi_{i+1/2,j,k} + \phi_{i-1/2,j,k}}{2}, \delta_x \phi_{i,j,k} = \frac{\phi_{i+1/2,j,k} - \phi_{i-1/2,j,k}}{h_x}, \nabla^2_{x,y}\phi_{i,j,k} = \delta^2_x \phi_{i,j,k} + \delta^2_y \phi_{i,j,k},$$
$$\{\phi\}^{\Omega_k} = \frac{1}{\Omega_k}\sum_{i,j}\phi_{i,j,k} h_x h_y, \quad \{\phi\}^V = \frac{1}{V}\sum_{i,j}\sum_{k=1}^{K_{i,j}}\phi_{i,j,k} h_z^k h_x h_y, \quad V = \sum_{i,j}\sum_{k=1}^{K_{i,j}} h_z^k h_x h_y. \tag{18}$$

Temperature, salinity, and horizontal velocity components are calculated at z_k horizons, vertical velocity is calculated at $z_{k+1/2}$ horizons. The distribution of variables is given in Fig. 1.

F i g. 1. Distribution of variables in box (i, j, k). PV (ω) is determined at the box vertices indicated by an asterisk, and the components of absolute vorticity $\xi^x_{i,j+1/2,k}, \xi^y_{i+1/2,j,k+1/2}, \xi^z_{i+1/2,j+1/2,k}$ are determined at its edges

According to notations (6), (7) and (18), we write out the finite-difference equations of model (1)–(5) (differential in time) [9, 14]:

$$\frac{du_{i+1/2,j,k}}{dt} - \left[v, \xi^z\right]_{i+1/2,j,k} + \left[w, \xi^y\right]_{i+1/2,j,k} = -\delta_x (E_{i+1/2,j/k} + P_{i+1/2,j,k}), \tag{19.1}$$

$$\frac{dv_{i,j+1/2,k}}{dt} + \left[u, \xi^z\right]_{i,j+1/2,k} - \left[w, \xi^x\right]_{i,j+1/2,k} = -\delta_y (E_{i,j+1/2,k} + P_{i,j+1/2,k}), \tag{19.2}$$

$$\frac{dw_{i,j,k+1/2}}{dt} - \left[u, \xi^y\right]_{i,j,k+1/2} + \left[v, \xi^x\right]_{i,j,k+1/2} = -\delta_z (E_{i,j,k+1/2} + P_{i,j,k+1/2}) + g\rho_{i,j,k+1/2}, \tag{19.3}$$

$$\delta_x u_{i,j,k} + \delta_y v_{i,j,k} + \delta_z w_{i,j,k} = 0, \tag{20}$$

$$\frac{dT_{i,j,k}}{dt} + \delta_x (u_{i,j,k} T_{i,j,k}) + \delta_y (v_{i,j,k} T_{i,j,k}) + \delta_z (w_{i,j,k} T_{i,j,k}) = 0, \tag{21}$$

$$\frac{dS_{i,j,k}}{dt} + \delta_x (u_{i,j,k} S_{i,j,k}) + \delta_y (v_{i,j,k} S_{i,j,k}) + \delta_z (w_{i,j,k} S_{i,j,k}) = 0, \tag{22}$$

$$\rho_{i,j,k} = \alpha T_{i,j,k} + \beta S_{i,j,k}. \tag{23}$$

In accordance with difference operators (18), the vorticity components (Fig. 1) have the following form:

$$\xi^x_{i,j+1/2,k+1/2} = \delta_y (w_{i,j+1/2,k+1/2}) - \delta_z (v_{i,j+1/2,k+1/2}),$$

$$\xi^y_{i+1/2,j,k+1/2} = \delta_z(u_{i+1/2,j,k+1/2}) - \delta_x(w_{i+1/2,j,k+1/2}), \qquad (24)$$
$$\xi^z_{i+1/2,j+1/2,k} = \delta_x(v_{i+1/2,j+1/2,k}) - \delta_y(u_{i+1/2,j+1/2,k}) + f^z_{j+1/2}.$$

From approximation (24) it follows that at points i + 1/2, j + 1/2, k + 1/2 (analogous to (12)) the following is fulfilled:

$$\delta_x\xi^x + \delta_y\xi^y + \delta_z\xi^z = 0. \qquad (25)$$

We assume that the terms in square brackets on the left side of equations (19.1)–(19.3) are written down in the form

$$[v,\xi^z]_{i+1/2,j,k} = \frac{1}{3}\left\{2\overline{v_{i+1/2,j,k}}^{xy}\,\overline{\xi^z_{i+1/2,j,k}}^{y} + \frac{\overline{v_{i+1,j,k}\xi^z_{i+1/2,j,k}}^{y} + \overline{v_{i,j,k}\xi^z_{i-1/2,j,k}}^{y}}{2}\right\},$$

$$[w,\xi^y]_{i+1/2,j,k} = \frac{1}{3}\left\{2\overline{w_{i+1/2,j,k}}^{xz}\,\overline{\xi^y_{i+1/2,j,k}}^{z} + \frac{\overline{w_{i+1,j,k}\xi^y_{i+1/2,j,k}}^{z} + \overline{w_{i,j,k}\xi^y_{i-1/2,j,k}}^{z}}{2}\right\},$$

$$[u,\xi^z]_{i,j+1/2,k} = \frac{1}{3}\left\{2\overline{u_{i,j+1/2,k}}^{xy}\,\overline{\xi^z_{i,j+1/2,k}}^{x} + \frac{\overline{u_{i,j+1,k}\xi^z_{i,j+1/2,k}}^{x} + \overline{u_{i,j,k}\xi^z_{i,j-1/2,k}}^{x}}{2}\right\}, \qquad (26)$$

$$[w,\xi^x]_{i,j+1/2,k} = \frac{1}{3}\left\{2\overline{w_{i,j+1/2,k}}^{yz}\,\overline{\xi^x_{i,j+1/2,k}}^{z} + \frac{\overline{w_{i,j+1,k}\xi^x_{i,j+1/2,k}}^{z} + \overline{w_{i,j,k}\xi^y_{i,j-1/2,k}}^{z}}{2}\right\},$$

$$\left[u,\xi^y\right]_{i,j,k+1/2} = \frac{1}{3}\left\{2\overline{u_{i,j,k+1.2}}^{xz}\,\overline{\xi^y_{i,j,k+1/2}}^{x} + \frac{\overline{u_{i,j,k+1}\xi^y_{i,j,k+1/2}}^{x} + \overline{u_{i,j,k}\xi^y_{i,j,k-1/2}}^{x}}{2}\right\},$$

$$\left[v,\xi^x\right]_{i,j,k+1/2} = \frac{1}{3}\left\{2\overline{v_{i,j,k+1/2}}^{yz}\,\overline{\xi^x_{i,j,k+1/2}}^{y} + \frac{\overline{v_{i,j,k+1}\xi^x_{i+1/2,j,k}}^{z} + \overline{v_{i,j,k}\xi^x_{i-1/2,j,k}}^{z}}{2}\right\}.$$

Finite-difference representation (26) ensures, in the case of two-dimensional divergence-free motion in (x, y), (x, z), (y, z) planes, the fulfillment of two quadratic conservation laws: kinetic energy and enstrophy (vorticity square), as well as antisymmetry property [9, 14].

Taking into account expression (24) and relations (26), we write down the equations for the components of the absolute vorticity – for ξ^x at $i,\ j+1/2,\ k+1/2$ point, for ξ^y at $i+1/2,\ j+1/2,\ k$ point, and for ξ^z at $i+1/2,\ j+1/2,\ k$ point (analogue of equation (13)):

$$\frac{d\xi^x}{dt} + \delta_y([v,\xi^x]) + \delta_z([w,\xi^x]) - \delta_y([u,\xi^y]) - \delta_z([u,\xi^z]) = \mathrm{g}\delta_y\overline{\rho}^z, \qquad (27.1)$$

$$\frac{d\xi^y}{dt} + \delta_x([u,\xi^y]) + \delta_z([w,\xi^y]) - \delta_x([v,\xi^x]) - \delta_z([v,\xi^z]) = -\mathrm{g}\delta_x\overline{\rho}^z, \qquad (27.2)$$

$$\frac{d\xi^z}{dt} + \delta_x([u,\xi^z]) + \mathrm{d}_y([v,\mathrm{x}^z]) - \delta_x([w,\xi^x]) - \delta_y([w,\xi^y]) = 0. \qquad (27.3)$$

We are to analyze the system of equations (27.1)–(27.3) in terms of fulfillment of conservation laws.

Let us consider, for example, two-dimensional divergence-free motion in (x, z) plane. In this case, continuity equation (20) is transformed to the form

$$\delta_x u_{i,j,k} + \delta_z w_{i,j,k} = 0 \tag{28}$$

and provides the introduction of the flow function: $u_{i+1/2,k} = \delta_z \psi^y_{i+1/2,k}$, $w_{i,k+1/2} = \delta_x \psi^y_{i,k+1/2}$.

From the continuity equation (28), it follows that

$$\xi^y_{i+1/2,\,k+1/2} = \delta_z^2 \psi^Y_{i+1/2,\,k+1/2} + \delta_x^2 \psi^y_{i+1/2,\,k+1/2}.$$

The vorticity equation is simplified and transformed [9] to the form

$$\frac{d\xi^y_{i+1/2,\,k+1/2}}{dt} + \frac{1}{3}\sum_{s=1}^{3} J_s(\psi^y_{i+1/2,\,k+1/2}, \xi^y_{i+1/2,\,k+1/2}) = 0, \tag{29}$$

where $J_1 = \delta_x(\overline{\psi^y_{i+1/2,\,k+1/2}}^x)\delta_y(\overline{\xi^y_{i+1/2,\,k+1/2}}^y) - \delta_y(\overline{\psi^y_{i+1/2,\,k+1/2}}^y)\delta_x(\overline{\xi^y_{i+1/2,\,k+1/2}}^x)$,

$$J_2 = \delta_x\left(\overline{\psi^y_{i+1/2,\,k+1/2}\ \delta_y(\overline{\xi^y_{i+1/2,\,k+1/2}}^y)}^x\right) - \delta_y\left(\overline{\psi^y_{i+1/2,\,k+1/2}\ \delta_x(\overline{\xi^y_{i+1/2,\,k+1/2}}^x)}^y\right), \tag{30}$$

$$J_3 = \delta_y\left(\overline{\xi^y_{i+1/2,\,k+1/2}\ \delta_x(\overline{\psi^y_{i+1/2,\,k+1/2}}^x)}^x\right) - \delta_x\left(\overline{\xi^y_{i+1/2,\,k+1/2}\ \delta_y(\overline{\psi^y_{i+1/2,\,k+1/2}}^y)}^x\right).$$

Expression (30) means that equation (29) has two quadratic invariants:

$$\frac{d}{dt}\sum_{i,k}\xi^y_{i+1/2,k+1/2} h_x h_z^k = 0, \quad \frac{d}{dt}\sum_{i,k}\left(\frac{(\delta_x\psi^y_{i+1/2,k+1/2})^2 + (\delta_z\psi^y_{i+1/2,k+1/2})^2}{2}\right) h_x h_z^k = 0,$$
$$\frac{d}{dt}\sum_{i,k}(\xi^y_{i+1/2,k+1/2})^2 h_x h_z^k = 0. \tag{31}$$

Approximation of nonlinear terms in equation (29) provides the antisymmetry property:

$$J(\psi^y_{i+1/2,\,k+1/2}, \xi^y_{i+1/2,\,k+1/2}) = -J(\xi^y_{i+1/2,\,k+1/2}, \psi^y_{i+1/2,\,k+1/2}). \tag{32}$$

A similar situation occurs for the motion in (y, z) and (x, y) planes:

$$\frac{d}{dt}\sum_{j,k}\xi^x_{j+1/2,k+1/2} h_y h_z^k = \frac{d}{dt}\sum_{j,k}\left(\frac{(\delta_y\psi^x_{j+1/2,k+1/2})^2 + (\delta_z\psi^x_{j+1/2,k+1/2})^2}{2}\right) h_y h_z^k =$$
$$= \frac{d}{dt}\sum_{j,k}(\xi^x_{j+1/2,k+1/2})^2 h_y h_z^k = 0, \tag{33}$$

$$J(\psi^x_{j+1/2,\,k+1/2}, \xi^x_{j+1/2,\,k+1/2}) = -J(\xi^x_{j+1/2,\,k+1/2}, \psi^x_{j+1/2,\,k+1/2}),$$

$$\frac{d}{dt}\sum_{i,j}\xi^z_{i+1/2,\,j+1/2} h_x h_y = \frac{d}{dt}\sum_{i,j}\left(\frac{(\delta_x\psi^z_{i+1/2,\,j+1/2})^2 + (\delta_y\psi^z_{i+1/2,\,j+1/2})^2}{2}\right) h_x h_y =$$
$$= \frac{d}{dt}\sum_{i,k}(\xi^z_{i+1/2,\,j+1/2})^2 h_x h_y = 0, \tag{34}$$

$$J(\psi^z_{i+1/2,\,j+1/2}, \xi^z_{i+1/2,\,j+1/2}) = -J(\xi^z_{i+1/2,\,j+1/2}, \psi^z_{i+1/2,\,j+1/2}).$$

Along with the feature of total energy conservation [5], relations (31)–(34) ensure the presence of two quadratic discrete invariants in the system of equations (1)–(5) with boundary (8)–(9) and initial (10) conditions.

Discrete equation of potential vorticity of a stratified incompressible fluid

We consider the properties of system (27.1)–(27.3) in the case of three-dimensional motion. We write this system in the following form:

$$\frac{d\overline{\xi^x_{i+1/2,\,j+1/2,k+1/2}}^x}{dt}+\overline{\Upsilon^x_{i+1/2,\,j+1/2,k+1/2}}^x=g\delta_y(\overline{\rho_{i+1/2,\,j+1/2,k+1/2}}^{xz}), \tag{35.1}$$

$$\frac{d\overline{\xi^y_{i+1/2,\,j+1/2,k+1/2}}^y}{dt}+\overline{\Upsilon^y_{i+1/2,\,j+1/2,k+1/2}}^y=-g\delta_x(\overline{\rho_{i+1/2,\,j+1/2,k+1/2}}^{yz}), \tag{35.2}$$

$$\frac{d\overline{\xi^z_{i+1/2,\,j+1/2,k+1/2}}^z}{dt}+\overline{\Upsilon^z_{i+1/2,\,j+1/2,k+1/2}}^z=0, \tag{35.3}$$

where

$$\begin{aligned}
&\Upsilon^x_{i,\,j+1/2,k+1/2}=\delta_y([v,\xi^x])_{i,\,j+1/2,k}+\delta_z([w,\xi^x])_{i,\,j+1/2,k}-\\
&-\delta_y([u,\xi^y])_{i,\,j+1/2,k}-\delta_z([u,\xi^z])_{i,\,j+1/2,k},\\
&\Upsilon^y_{i+1/2,\,j,k+1/2}=\delta_x([u,\xi^y])_{i+1/2,\,j,k+1/2}+\delta_z([w,\xi^y])_{i+1/2,\,j,k+1/2}-\\
&-\delta_x([v,\xi^x])_{i+1/2,\,j,k+1/2}-\delta_z([v,\xi^z])_{i+1/2,\,j,k+1/2},\\
&\Upsilon^z_{i+1/2,\,j+1/2,k}=\delta_x([u,\xi^z])_{i+1/2,\,j+1/2,k}+\delta_y([v,\xi^z])_{i+1/2,\,j+1/2,k}-\\
&-\delta_x([w,\xi^x])_{i+1/2,\,j+1/2,k}-\delta_y([w,\xi^y])_{i+1/2,\,j+1/2,k}.
\end{aligned} \tag{36}$$

It is easy to verify that according to relation (25), equations (27.1)–(27.3) at point i + 1/2, j + 1/2, k + ½ and definition (36) we obtain

$$\delta_x(\Upsilon^x_{i+1/2,\,j+1/2,k+1/2})+\delta_y(\Upsilon^y_{i+1/2,\,j+1/2,k+1/2})+\delta_z(\Upsilon^z_{i+1/2,\,j+1/2,k+1/2})=0. \tag{37}$$

Note that equality (37) is satisfied because density linearly depends on temperature and salinity. We assume that from equations (21)–(23) follows a discrete equation for density, which has the following form:

$$\frac{d\rho_{i,j,k}}{dt}+\delta_x(u_{i,j,k}\rho_{i,j,k})+\delta_y(v_{i,j,k}\rho_{i,j,k})+\delta_z(w_{i,j,k}\rho_{i,j,k})=0. \tag{38}$$

We write equation (38) at the points $(i,\ j+1/2,k+1/2)$, $(i+1/2,\ j,k+1/2)$, $(i+1/2,\ j+1/2,k)$, respectively:

$$\begin{aligned}
&\frac{d\overline{\rho_{i,\,j+1/2,k+1/2}}^{yz}}{dt}+\delta_x(\overline{u_{i,\,j+1/2,k+1/2}\rho_{i,\,j+1/2,k+1/2}}^{yz})+\delta_y(\overline{v_{i,\,j+1/2,k+1/2}\rho_{i,\,j+1/2,k+1/2}}^{yz})+\\
&+\delta_z(\overline{w_{i,\,j+1/2,k+1/2}\rho_{i,\,j+1/2,k+1/2}}^{yz})=0,
\end{aligned} \tag{39.1}$$

$$\begin{aligned}
&\frac{d\overline{\rho_{i+1/2,\,j,k+1/2}}^{xz}}{dt}+\delta_x(\overline{u_{i+1/2,\,j,k+1/2}\rho_{i+1/2,\,j,k+1/2}}^{xz})+\delta_y(\overline{v_{i+1/2,\,j,k+1/2}\rho_{i+1/2,\,j,k+1/2}}^{xz})+\\
&+\delta_z(\overline{w_{i+1/2,\,j,k+1/2}\rho_{i+1/2,\,j,k+1/2}}^{xz})=0,
\end{aligned} \tag{39.2}$$

$$\begin{aligned}
&\frac{d\overline{\rho_{i+1/2,\,j+1/2,k}}^{xy}}{dt}+\delta_x(\overline{u_{i+1/2,\,j+1/2,k}\rho_{i+1/2,\,j+1/2,k}}^{xy})+\delta_y(\overline{v_{i+1/2,\,j+1/2,k}\rho_{i+1/2,\,j+1/2,k}}^{xy})+\\
&+\delta_z(\overline{w_{i+1/2,\,j+1/2,k}\rho_{i+1/2,\,j+1/2,k}}^{xy})=0.
\end{aligned} \tag{39.3}$$

We differentiate equation (39.1) in the finite-difference sense with respect to *x*, (39.2) with respect to *y* and (39.3) – to *z*. As a result, we get

$$\frac{d\delta_x(\overline{\rho}^{yz}_{i+1/2,j+1/2,k+1/2})}{dt}+\delta_x(R^x_{i+1/2,j+1/2,k+1/2})=0, \tag{40.1}$$

$$\frac{d\delta_y(\overline{\rho}^{xz}_{i+1/2,j+1/2,k+1/2})}{dt}+\delta_y(R^y_{i+1/2,j+1/2,k+1/2})=0, \tag{40.2}$$

$$\frac{d\delta_z(\overline{\rho}^{xy}_{i+1/2,j+1/2,k+1/2})}{dt}+\delta_z(R^z_{i+1/2,j+1/2,k+1/2})=0, \tag{40.3}$$

where the notations $R^x_{i,j+1/2,k+1/2}, R^y_{i+1/2,j,k+1/2}, R^z_{i+1/2,j+1/2,k}$ are obvious.

Let us mention the properties of the introduced functions in equations (39.1)–(39.3) and (40.1)–(40.3). They have the following form:

$$\overline{\overline{\rho}^{yz}_{i,j+1/2,k+1/2}}^{x}=\overline{\overline{\rho}^{xz}_{i+1/2,j,k+1/2}}^{y}=\overline{\overline{\rho}^{xy}_{i+1/2,j+1/2,k}}^{z}=\overline{\rho_{i+1/2,j+1/2,k+1/2}}^{xyz}, \tag{41}$$

$$\overline{R^x_{i,j+1/2,k+1/2}}^{x}=\overline{R^y_{i+1/2,j,k+1/2}}^{y}=\overline{R^z_{i+1/2,j+1/2,k}}^{z}=\overline{\delta_x(u_{i+1/2,j+1/2,k+1/2}\rho_{i+1/2,j+1/2,k+1/2})+}^{xyz}$$
$$\overline{+\delta_y(v_{i+1/2,j+1/2,k+1/2}\rho_{i+1/2,j+1/2,k+1/2})+\delta_z(w_{i+1/2,j+1/2,k+1/2}\rho_{i+1/2,j+1/2,k+1/2})}^{xyz}.$$

We assume that the difference analogue of PV is defined at point $i+1/2,\ j+1/2,\ k+1/2$ and is written down as follows: $\varpi_{i+1/2,j+1/2,k+1/2}=$

$$=\overline{\xi^x_{i+1/2,j+1/2,k+1/2}}^{x}\delta_x(\overline{\rho}^{yz}_{i+1/2,j+1/2,k+1/2})+\overline{\xi^y_{i+1/2,j+1/2,k+1/2}}^{y}\delta_y(\overline{\rho}^{xz}_{i+1/2,j+1/2,k+1/2})+$$
$$+\overline{\xi^z_{i+1/2,j+1/2,k+1/2}}^{z}\delta_z(\overline{\rho}^{xy}_{i+1/2,j+1/2,k+1/2}). \tag{42}$$

Determination of potential vorticity of a three-dimensional stratified fluid (42) at the box vertices is determined by the fulfillment of discrete equation (25).

To obtain a difference analogue of Ertel's theorem (15), we multiply equations (35.1)–(35.3) by $\delta_x(\overline{\rho}^{yz}_{i+1/2,j+1/2,k+1/2})$, $\delta_y(\overline{\rho}^{xz}_{i+1/2,j+1/2,k+1/2})$, $\delta_z(\overline{\rho}^{xy}_{i+1/2,j+1/2,k+1/2})$, and the system (40.1)–(40.3) – by $\overline{\xi^x_{i+1/2,j+1/2,k+1/2}}^{x}, \overline{\xi^y_{i+1/2,j+1/2,k+1/2}}^{y}, \overline{\xi^z_{i+1/2,j+1/2,k+1/2}}^{z}$, respectively. As a result, at point *i* + 1/2, *j* + 1/2, *k* + 1/2 we get

$$\frac{d\omega}{dt}+\overline{\Upsilon^x}^{x}\delta_x(\overline{\rho}^{yz})+\overline{\Upsilon^y}^{y}\delta_y(\overline{\rho}^{xz})+\overline{\Upsilon^z}^{z}\delta_z(\overline{\rho}^{xy})+\overline{\xi^x}^{x}\delta_x R^x+\overline{\xi^y}^{y}\delta_y R^y+\overline{\xi^z}^{z}\delta_z R^z=0.$$

Since ξ^x,ξ^y,ξ^z satisfy relation (25), $\Upsilon^x,\Upsilon^y,\Upsilon^z$ – relation (37), with regard to equalities (41), at point *i* + 1/2, *j* + 1/2, *k* + 1/2 we obtain finite-difference equation of potential vorticity (differential in time) of a stratified incompressible fluid in a divergent form:

$$\frac{d\omega}{dt}+\delta_x(\Upsilon^x\overline{\rho}^{yz}+\xi^x R^x)+\delta_y(\Upsilon^y\overline{\rho}^{xz}+\xi^y R^y)+\delta_z(\Upsilon^z\overline{\rho}^{xy}+\xi^z R^z)=0. \tag{43}$$

Note that formally there is no difficulty in obtaining equation (43) with temporal discretization. Due to additional indexing, difference equations become cumbersome and therefore difficult to read.

Strictly speaking, the form of nonlinear terms in equation (43) does not correspond to their differential analogue (15). The integral over domain from $\frac{d\omega}{dt}$ is equal to zero when the following relations are satisfied at the boundaries:

$$\Upsilon^x_{i,j+1/2,k+1/2} = 0, \quad R^x_{i,j+1/2,k+1/2} = 0 \text{ at } \quad i = i_1, \quad i = i_N,$$
$$\Upsilon^y_{i+1/2,j,k+1/2} = 0, \quad R^y_{i+1/2,j,k+1/2} = 0 \text{ at } \quad j = j_1, \quad j = j_M, \qquad (44)$$
$$\Upsilon^z_{i+1/2,j+1/2,k} = 0, \quad R^z_{i+1/2,j+1/2,k} = 0 \text{ at } \quad k = 1/2, \quad k = K_{i,j} + 1/2,$$

From relations (44) it follows that additional boundary conditions, which are absent in the original formulation, are required.

Conclusion

A finite-difference analogue of absolute vorticity is written out for the system of equations of an ideal fluid without a quasi-static approximation. Projections of this equation onto two-dimensional subspaces (x, y), (y, z), (x, z) preserve energy, vorticity, enstrophy and have the antisymmetry property. To obtain the well-known Arakawa–Lamb scheme for the shallow water equations, it is necessary to write out an original difference system of equations for horizontal velocities that differs from (26).

The original result is the obtained discrete equation for potential vorticity of a stratified incompressible fluid as an exact consequence of the original finite-difference system of equations which have a divergent form. In this case, density satisfied the linear equation of state, the approximation of which in this case ensures both total energy conservation and the divergent form of equation for PV. If a nonlinear density dependence on temperature and salinity is applied, a special density approximation at the box edges is required to preserve total energy. In this case, an additional term occurs in the discrete potential vorticity equation; it has no analogue in the differential problem. Another problem arises from the form of advective terms in the PV equation, which are fundamentally different from their differential counterparts. In order for the potential vorticity to be conserved, we require additional boundary conditions under which the PV discrete analogue is an invariant.

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