Nonlinear Effects at Propagation of Long Surface Waves in the Channels with a Variable Cross-Section

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Analysis of the propagation of waves in the channels and straits of different geometry seems an important problem, since the change of the channel width, its depth and cross-sectional shape in general significantly affects the amplification and weakening of the spatial structure of the wave field. Propagation and deformation of single surface waves in the channels with a variable cross-section are analyzed within the framework of shallow-water equations. It is shown that nonlinearity is manifested in growth with time of the wave front slope steepness and its subsequent breaking. Height and length of a propagating wave are weakly influenced by nonlinearity. The distance traversed by a single wave up to its breaking decreases with growth of the wave height and diminution of its length. The wave amplitude characteristics are estimated depending on the channel depth and width. They are accurately described by the known Green's law. The propagation of long waves in channels with different cross-sectional shape at the same maximum depth and width had been numerically analyzed. The strongest nonlinearity was detected in the channel with a triangular cross section, as in this case the cross-sectional area was the least with the same depth and width. The channel shape has no significant effect on the amplitude and wave characteristics.

Keywords: waves in fluid, waves of finite amplitude, wave propagation in channels, channel mathematical model, numerical solutions, manifestation of nonlinear effects.

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Introduction. Propagation of long waves in the channels, bays, straits and other irregularities of the coastline may be accompanied by their considerable intensification. Nonlinear long waves in the basins with varying depth and deformation of their profiles during propagation in shallow water were studied numerically and analytically in many papers, in [1 – 11], in particular. It seems to be important to analyze propagation of waves in the channels and straits of different geometry, since, on the whole, change of the channel width, depth and the cross-section form significantly affects intensification, weakening and spatial structure of a wave field. This problem was studied in the linear approximation, for example, in [1 – 3, 11 – 14]. In a number of papers considering nonlinear waves in the channels [15 – 18] the features of propagation of such waves are investigated including transformation of their forms and variation of the heights. In [19] the waves are calculated by the characteristics method for the channel the shelf zone width of which varies according to the linearity law.

The present paper considers propagation of single nonlinear long waves in the straight channels with a variable cross-section. Evolution of such waves along the channel is analyzed numerically depending on variation of the cross-section geometry. The problem on the wave propagation was solved numerically within the framework of the shallow water equations of long waves [20] in which the horizontal velocity and displacement of the fluid free surface averaged across the channel are used as the basic variables

Special attention is paid to the influence of the channel cross-section geometry upon the wave propagation character, its amplitude characteristics and form.

Mathematical formulation of the problem. In the horizontal plane *Oxy* (*x*, *y* are the Cartesian coordinates) considered is the propagation of long waves in the channel of variable cross-section the depth of which $z = -h\ (x)$ (z is the vertical coordinate) and the mirror cross-section covers the area $|\,y\,| \le b(x)$. The plane $z = 0$ coincides with the fluid undisturbed (horizontal) surface. Within the framework of the channel theory for non-linear long waves, considered is propagation of a single wave in such a channel when it enters the channel through the cross-section $x = 0$ and leaves it through the cross-section $x = L$. The wave initial height $a_0 = 1$ m.

The equations of the non-linear theory of long waves include the horizontal velocity averaged over the cross section $u = u\ (x, t)$ (t is time) and displacement of the fluid free surface averaged along the cross-section of the channel mirror $\zeta =$ $= \zeta\ (x, t)$. The equations of fluid motion in such channels can be obtained by integrating the equations of non-linear theory of long waves over the cross-section area [15, 16]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial H}{\partial x} = g\frac{dh}{dx}, \qquad \frac{\partial S}{\partial t} + \frac{\partial}{\partial x}\left(Su\right) = 0, \tag{1}$$

where g is free fall acceleration; $S(x, t)$ is a dynamic area of the channel vertical cross-section; $H(x, t) = h(x) + (x, t)$ is a full depths on the channel longitudinal axis.

Let us consider the channel the cross-sections of which are symmetrical relative to the vertical strait line *Oz*, namely $z(y) = -H + \alpha(x)\left|y\right|^m$ [16], at that in contrast to [16], the parameter α is considered to be a variable function x. Parameter $m \in (0, +\infty)$ characterizes the channel cross-section form. If $m = 1$ the cross-section is triangular, at $m = 2$ it is parabolic, when $m \to +\infty$ the cross-section form tends to be rectangular. The variable coefficient $\alpha\ (x)$ can be defined if one assumes that at the undisturbed state $\zeta = 0$ the mirror local semi-width in the channel is equal to $b(x)/2$. Thus $\alpha(x) = 2^m h / b^m$. Having defined the integration limits on y in the cross-section $z(y) = -H + \alpha(x)\left|y\right|^m$, we can find its dynamic area:

$$S = 2\int_0^{\sqrt[m]{H/h}\cdot b/2} \left(-H + \frac{2^m h}{b^m}\left|y\right|^m\right)dy = \frac{m}{m+1}\frac{bH^{m+1/m}}{h^{1/m}}. \tag{2}$$

As ζ is rather small let us consider the mirror width $b = b(x)$ to be independent on time t. Then after inserting (2) to (1) we obtain the equation system closed relative to u and H:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \zeta}{\partial x} = 0\,, \tag{3}$$

$$\frac{\partial H}{\partial t} + \frac{mH}{(m+1)b}u\frac{db}{dx} + u\frac{\partial H}{\partial x} - \frac{H}{(m+1)h}u\frac{dh}{dx} + \frac{mH}{m+1}\frac{\partial u}{\partial x} = 0\,. \tag{4}$$

At $m \to +\infty$ equation (4) is transformed into the continuity equation for the waves propagating in the channel with a rectangular section, but with variable depth and width. As a result the equation system (3), (4) takes the following form [21]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial H}{\partial x} = g\frac{dh}{dx}, \quad \frac{\partial H}{\partial t} + \frac{1}{b}\frac{\partial (Hbu)}{\partial x} = 0\ . \tag{5}$$

The initial conditions for the equation system (3), (4) require presetting of the fields u and ζ at the initial moment of time. Let us assume that at $t = 0$ the channel fluid is in an undisturbed state, i. e.

$$u(x,0) = \zeta(x,0) = 0\,. \qquad (6)$$

The boundary conditions simulate entering of a single wave to the channel through the left boundary and its free exiting the channel either through the right boundary or through the left one. The conditions of the wave exiting the channel start their functioning from the moment of its complete entering the channel through the left boundary:

$$\zeta = a_0 \sin(\pi\tau_1), \quad u = \sqrt{\frac{gb}{S}}\zeta \quad (x = 0,\ 0 \le t \le T_1)\ , \qquad (7)$$

$$\frac{\partial u}{\partial t} - C_0 \frac{\partial u}{\partial x} = 0 \ \ (x = 0,\ t \ge T_1)\,, \qquad (8)$$

$$\frac{\partial u}{\partial t} + C_1 \frac{\partial u}{\partial x} = 0 \ \ (x = L,\ t \ge 0)\,, \qquad (9)$$

where $\tau_1 = t / T_1$; $T_1 = \lambda / C_0$; λ is the wave length at the channel entrance; $C_0 = C(0)$; $C_1 = = C(L)$.

Within the framework of the equation system (3), (4) the expression for the wave propagation velocity is written in the form [7, 15]:

$$V(\zeta) \approx \sqrt{gh}\left[1 + \left(\frac{3}{2} + \frac{1}{m}\right)\frac{\zeta}{h}\right]. \qquad (10)$$

Derivative $\partial\zeta/\partial x$ grows unlimitedly during the time

$$t^* = -\frac{1}{\left.\left(\dfrac{dV}{d\zeta}\dfrac{d\zeta_0}{dx}\right)\right|_{\max}}\,. \qquad (11)$$

The distance traversed by a wave and characterizing the area of non-linear effects' manifestation corresponds to this time period:

$$L^* = Vt^* = \frac{2h}{3\left|d\zeta_0 / dx\right|_{\max}}\,. \qquad (12)$$

For a single sinusoidal elevation in a semi-wave form (7) we find distance Λ after passing of which the wave breaks [7]:

$$\Lambda = \frac{2m}{(3m+2)}\frac{\lambda h}{\pi\zeta_0}\,. \qquad (13)$$

Propagation of nonlinear waves in the basins of the model geometry. Within the framework of the non-linear theory of long waves the process of a single wave deformation at its propagation in the basin with constant depths is studied. It is shown in Fig. 1. Note that the local horizontal velocity of the wave current varies proportionally to the free surface local displacement. In the linear approximation, the free surface profile retains its form (curve *1*).

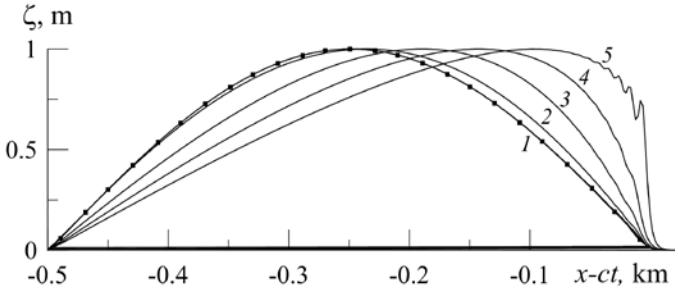

Fig. 1. Evolution of a single surface wave (the initial height is 1 m and the wavelength is 0.5 km) in the basin of 25 m depth: curve *1* is a linear wave; curves *2* – *5* are the non-linear waves at the time moments 30, 60, 90, 120 sec, respectively

Analysis of the obtained dependences shows that influence of non-linearity is manifested in increase with time of the free surface steepness on the wave front slope that, in its turn, results in a subsequent wave breaking. The wave breaking start is shown by curve 5.

Transformation of a single long wave at its propagation over the sloping bottom with a constant slope 1° is also investigated. The basin depth decreases linearly from 500 to 10 m when a wave passes the distance 56 km. The initial wavelength is 5 km.

Fig. 2 shows the wave profiles and distributions of the horizontal velocity. When a wave comes to shallow water its propagation velocity decreases and its maximum height grows almost twice.

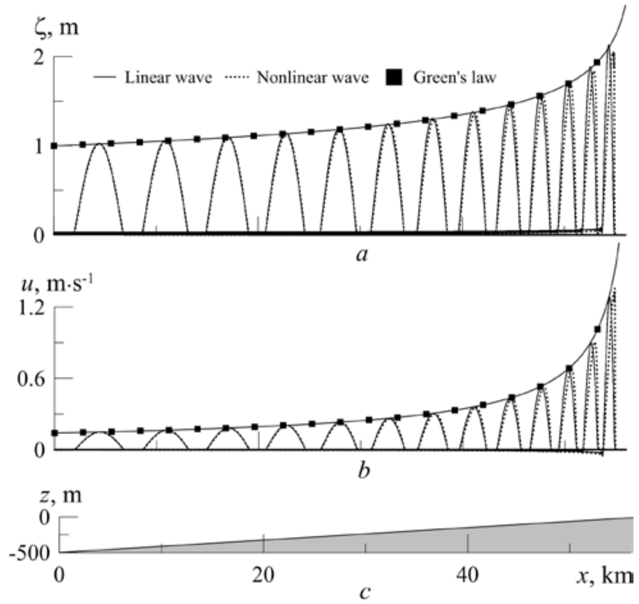

Fig. 2. Transformation of the wave form (*a*) and the horizontal velocity distribution (*b*) with time in the zone over the sloping bottom (*c*). The initial wave height is 1 m, the wavelength is 5 km

Distribution of the amplitudes' fields in the linear approximation is governed by the Green law [20] according to which the wave height is proportional to $h^{-1/4}$

in the preset point *x*. The amplitude of the wave velocity increases while approaching the coastline according to another law, $h^{-3/4}$. Thus, maximum wave height and wave velocity should be described by the follow relations, respectively:

$$A_{\max}(x) = a_0 \frac{h_0^{1/4} b_0^{1/2}}{h(x)^{1/4} b(x)^{1/2}}, \quad U_{\max}(x) = \left(\frac{gb(x)}{S(x)} \right)^{1/2} A_{\max}(x). \qquad (14)$$

Fig. 2 *a*, *b* represents analytical assessments of the amplitude characteristics of the waves and current velocity calculated by the Green law. The analogous dependences of the wave heights upon the channel local geometry are obtained and discussed in [12 – 14]. At that the wavelength, as it is expected, decreases proportionally to $h^{1/2}$ [20]. When a wave comes to shallow water its tail represents an extended wave of elevation. The wave current velocity and that of a leading wave are directed in opposite directions. The nonlinear wave is characterized by small decrease of height (Fig. 2, *a*) and horizontal velocity (Fig. 2, *b*) with time.

Let us consider the effect of the channel width change on propagation of a non-linear long wave in a straight channel of constant depth h = 100 m. The width of the middle part of the channel varies with distance linearly, i. e. locally narrows (*b* (*x*) = 1000 m, 0 <*x* < 20 km; *b* (*x*) = 400 m, 40 <*x* <60 km) or locally expands (*b* (*x*) = 400 m, 0 <*x* <<20 km; *b* (*x*) = 1000 m, 40 <*x* < 60 km). At the wave propagation (the wavelength is 2 km) on the part where the channel expands, the wave height and the wave current horizontal velocity decrease by 1.5 times (Fig. 3 *a*, *b*.), whereas on the part where the channel narrows – increase in a similar way (Fig. 4 *a*, *b*).

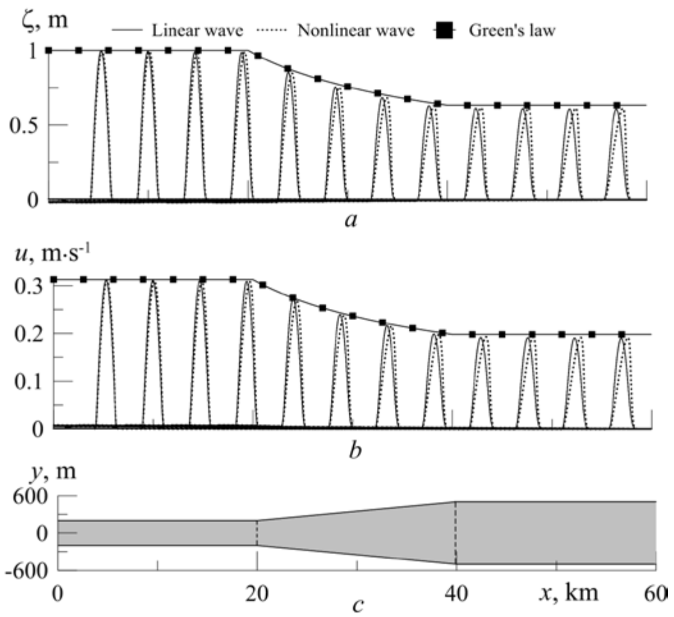

Fig. 3. Transformation of the wave form (*a*) and the horizontal velocity distribution (*b*) with time for a widening channel (*c*) at increase of its width from 400 to 1000 m. The channel depth is constant and equals 100 m, the wavelength is 2 km

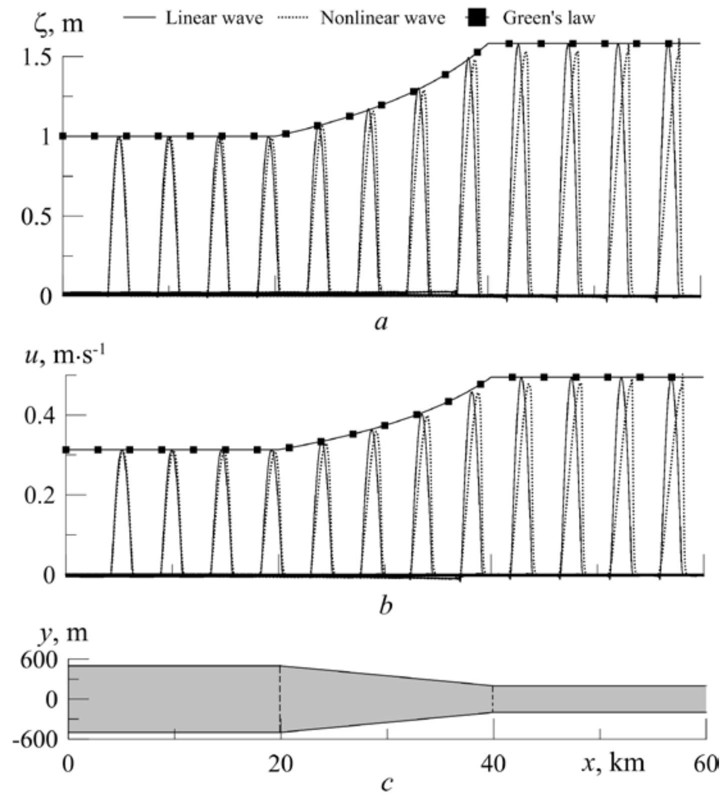

Fig. 4. Transformation of the wave form (*a*) and the horizontal velocity distribution (*b*) with time for a narrowing channel (*c*) at decrease of its width from 1000 to 400 m

The amplitude characteristics of the fields vary proportionally to $b^{-1/2}$. These dependences in a form of the envelopes (analogs of the Green law) are shown in Fig. 3 *a*, *b*. The analogous dependences were observed during propagation of waves in the straits of arbitrary cross-section [12 – 14]. The non-linear effects are manifested in the following way: the wave profile is deformed, the front slope becomes steeper, the wave crest moves forward up to wave braking.

The computation experiments aimed at comparing the processes of single linear and non-linear waves' evolution (Fig. 2 – 4) showed that no significant changes of a wave length, height and horizontal velocity are observed, though nonlinearity affects increase with time of the free surface profile steepness on the wave front slope.

Besides, propagation of long waves in the channels with different cross-section forms the areas of which are calculated by formula (2) is analyzed. The channel cross-section may be triangular (at $m = 1$), parabolic (at $m = 2$) and rectangular (when $m \to +\infty$).

Increase of a wave steepness leads to growth of the vertical velocity projection up to the moment when the conditions of applicability of the long waves' equation system are violated. Let us estimate the distance which the wave passed from the channel entrance to the wave breaking.

Fig. 5 shows (in the logarithmic scale) dependence of a wave breaking length Λ normalized to the wavelength λ, upon the non-linearity parameter a_0/h. Numerical calculations are performed for various channel depths at the following wave parameters: a_0 = 1 m, λ = 2 km. Curves 1 – 3 are calculated for different forms of the channel cross-section. The estimates obtained from the analytical formula (13) are also represented here. Numerical and analytical calculations are in good agreement.

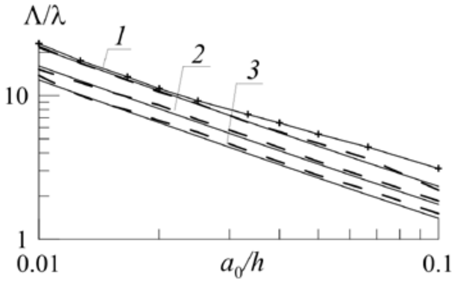

Fig. 5. Dependences of the distance related to the initial wavelength (*Λ*/ λ) and traversed by a non-linear wave up to its breaking, upon the non-linearity parameter a_0/h for different forms of the channel cross-section: *1* – at $m = +\infty$; *2* – at $m = 2$; *3* – at $m = 1$ (dash line denotes numerical estimates, solid line – analytical estimates, crosses – numerical estimates taking into account bottom friction)

For the waves propagating in the channel of a rectangular cross-section, influence of bottom friction on the parameters of a non-linear wave was assessed. The friction was preset in the motion equation (3) by the additional item $ku\,|\,u\,|\,/(h+\zeta)$, where $k = 2.6\cdot10^{-3}$. It is seen in Fig. 5 that as the wave initial height grows and the channel depth decreases, the bottom friction influence upon the distance traversed by the wave up to its breaking increases.

The highest value of the wave breaking length is achieved in the channel with a rectangular cross-section ($m \to +\infty$), the smallest one – in the channel with a triangular cross-section ($m = 1$). It is related to the fact that when $m \to +\infty$ the channel cross-section area takes on the maximum possible value at the given depth h and the mirror width b; whereas when m → 1 it becomes minimal. All the curves are qualitatively similar: the wave of large amplitude breaks almost immediately at the moment when it was formed. At the same time when the amplitude is small the breaking length is inversely-proportional to the wave amplitude and can be large enough for a wave not to break on a large distance.

When the channel depth decreases, non-linearity influence upon the wave amplitude characteristics increases. The wave front slope steepness grows faster; it results in its earlier breaking. Besides, the wave propagation velocity in the deeper channel grows.

Dependence of the free surface amplitudes and the wave velocity upon the channel geometry was investigated. Fig. 6 shows the dependences of the maximum amplitude A_{max} normalized to a_0 (Fig. 6 *a*, *c*) and the maximum horizontal velocity U_{max} normalized to u_0 (Fig. 6 *b*, *d*) upon the basin depth h/h_0 (Fig. 6, *a*, *b*) and the channel width b/b_0 (Fig. 6, *c*, *d*) at constant depth. In a linear case (curves *1*) the amplitudes and the wave velocities do not depend on the channel form and vary according to the Green law. The non-linear waves (curves *2*) propagate almost without changing their amplitude and horizontal velocity. The asterisks in Fig. 6 separate the broken (on the left of them) and unbroken (on the right of them) waves. When the bottom friction is taken into consideration (curves *3*) the wave heights decrease somewhat faster, and the distance traversed by a wave before breaking slightly grow.

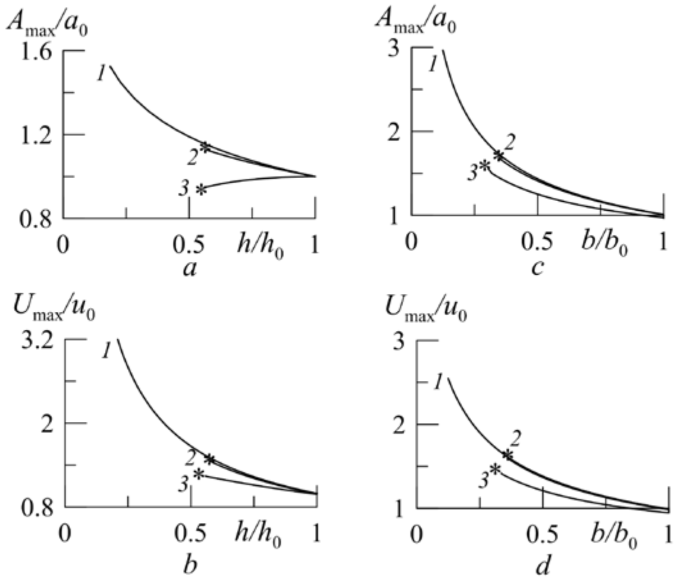

Fig. 6. Dependences of the dimensionless wave amplitudes A_{max}/a_0 (*a*, *c*) and dimensionless wave velocities U_{max}/u_0 (*b*, *d*) upon the basin depth h/h_0 (*a*, *b*) and the channel width b/b_0 (*c, d*) at constant depth 100 m for linear (curves *1*) and non-linear (curves *2*) waves, and also at taking into account bottom friction (curves *3*)

Conclusions. The numerical analysis shows that non-linearity affects increase with time of the free surface profile steepness of the wave front slope. Being influenced by non-linearity, the wavelength and height are reduced insignificantly.

When a wave propagates of the basin with constant depth observed is the increase of steepness of the wave front slope free surface and its subsequent breaking, at that the wave amplitude slightly decreases. A linear wave propagates without changing its height and current horizontal velocity.

When a wave comes to shallow water in the basin with a linearly decreasing depth it intensifies**,** its horizontal velocity grows and the wavelength decreases.

When a wave propagates in the channel with local extension or local narrowing (the depth remains constant), in the extending parts the free surface heights and the horizontal velocity amplitudes diminish by 1.5 times, whereas on the narrowing parts they increase by the same value. As it was expected, the wavelength remained unchanged.

Within the framework of the channel theory for long waves no significant difference between the amplitude characteristics of linear and non-linear waves were revealed. For all the results obtained numerically the linearity effect is manifested in changing of the wave form, namely, in increase of steepness of the wave front slope free surface, small decrease of the wave height followed by its subsequent breaking.

Propagation of long waves in the channels with different cross-section forms at one and the same maximum depth and mirror width is numerically analyzed. The strongest manifestation of nonlinearity is revealed for the channel with a triangular cross-section since in such a case the section area turns out to be the smallest at one and the same depth and mirror width. The channel form produces no considerable effect upon the amplitude and wave characteristics.

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