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Original article

Vertical Momentum Transfer Due to Internal Waves

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Abstract
Purpose. The work aims to study the vertical momentum transfer by internal waves at the exit of the Strait of Gibraltar into the Mediterranean Sea, accounting for turbulent viscosity and diffusion.
Methods and Results. In contrast to the traditional approach relating vertical momentum transfer to small-scale turbulence, the present study examines the wave transport mechanism. The wave field is described using classical hydrodynamic equations for a stratified incompressible fluid with shear flow, incorporating turbulent viscosity and diffusion. The boundary value problem for the vertical velocity amplitude of internal waves, which conditions the mode structure, is solved numerically. In the linear approximation, the complex nature of the coefficients results in a complex solution, leading to a non-zero vertical wave momentum flux. The impact of horizontal turbulent viscosity and diffusion on this flux is investigated. Three models are compared: the first one – with constant exchange coefficients (basic case), the second – with exchange coefficients depending on phenomenon scale according to the “4/3” law, and the third – with coefficients of horizontal exchange taking into account stratification. It is shown that when the dependence of exchange coefficients on the phenomenon scale according to the “4/3” law is taken into account, the momentum flux is higher in magnitude than that with constant coefficients, but lower than the fluxes taking into account stratification. The same pattern holds for the vertical component of the Stokes drift velocity. The choice of exchange coefficients has virtually no effect on the horizontal component of the Stokes drift velocity.
Conclusions. The dispersion curves of internal waves are independent of the choice of exchange coefficients. However, the wave attenuation decrement is sensitive to this choice: it is higher in magnitude when the exchange coefficients depend on the phenomenon scale according to the “4/3” law compared to the case of constant exchange coefficients, and even higher in absolute value when stratification is taken into account. The same pattern holds true for the vertical wave momentum flux.

Keywords: internal waves, wave momentum flux, Stokes drift, turbulent viscosity, turbulent diffusion

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Introduction

Vertical momentum transport is typically attributed to small-scale turbulence generated by wind, currents and the breaking of surface and internal waves [1] [1–8]. In addition to internal wave breaking, a “gentle” regime sustains turbulence previously generated by velocity shear within an internal wave [9]. Similarly, in

[1] Monin, A.S. and Ozmidov, R.V., 1985. *Turbulence in the Ocean*. Environmental Fluid Mechanics Series. Dordrecht, Boston, Lancaster: D. Reidel Publishing Company, 247 p.

shear currents with near-critical Richardson numbers, the flow may sustain turbulence without breaking [10]. In the bottom layer, small-scale turbulence can arise within the bottom boundary layer due to the interaction of currents and tides with irregular bottom topography [11]. Notably, three-dimensional internal waves can be trapped by a sloping bottom when the bottom slope aligns the group velocity vector of the reflected wave parallel to the bottom, converting the energy of the wave into turbulence [12].

In shear currents at critical layers, where the current velocity matches the phase velocity of internal waves, "cat's eye" vortex structures can emerge [13, 14]. Small-amplitude internal waves are significantly attenuated by small-scale turbulence, whereas large-amplitude waves experience minimal attenuation but can enhance turbulence [15, 16]. As internal waves propagate through a horizontally inhomogeneous ocean into regions of shallower depths, their amplitude increases, leading to pronounced nonlinear effects and energy dissipation into turbulence. A similar mechanism is observed in a horizontally inhomogeneous waveguide: internal wave propagation results in wave trapping and focusing, leading to energy dissipation into turbulence [17]. Internal waves often propagate as wave packets [18, 19]. During the propagation of weakly nonlinear internal wave packets, second-order mean currents are induced relative to the wave amplitude [20, 21]. At the leading and trailing edges of these wave packets, the vertical component of the induced current has opposite signs, resulting in no vertical transport.

Internal waves, even without breaking, contribute to vertical exchange in the ocean. In a dissipative medium with viscosity and diffusion, internal waves undergo attenuation [22–24]. When turbulent viscosity and diffusion are considered, the phase shift between vertical and horizontal velocity oscillations deviates from π/2, resulting in non-zero vertical wave momentum flux [25–31]. This arises because the eigenfunction equation for internal waves includes complex coefficients, yielding a complex solution to the corresponding boundary value problem [32, 33], with the wave frequency exhibiting a small imaginary part [2] [34–39]. Historically, horizontal turbulent exchange coefficients were assumed to be constant and independent of the phenomenon's scale [27–31]. However, work [26] accounted for the dependence of the horizontal turbulent exchange coefficient on the phenomenon's scale according to the "4/3" law [3, 4]:

$$M = c_1 \cdot l^{4/3}. \tag{1}$$

The c_1 proportionality coefficient in this law is independent of stratification, and a special case is considered where the wave propagates perpendicular to the flow, enabling an analytical solution with a constant buoyancy frequency.

This study utilizes real stratification and current profiles derived from field experiment data at the exit of the Strait of Gibraltar into the Mediterranean Sea.

[2] Vorotnikov, D.I., 2024. [*Transport Processes Caused by Inertial-Gravity Internal Waves*]. Thesis Cand. Phys.-Math.Sci. Moscow, 108 p. (in Russian).

[3] Ozmidov, R.V., 1968. [*Horizontal Turbulence and Turbulent Exchange in the Ocean*]. Moscow: Nauka, 200 p. (in Russian).

[4] Ozmidov, R.V., 1986. [*Diffusion of Impurities in the Ocean*]. Leningrad: Gidrometeoizdat, 278 p. (in Russian).

It incorporates the dependence of the horizontal exchange coefficient on the phenomenon's scale via the "4/3" law, with the c_1 proportionality coefficient linked to the Brunt–Väisälä frequency, based on drifter experiment data [40, 41]. These data indicate that the horizontal exchange coefficient is proportional to the product of the velocity scale *V* and the length scale *L*, where the length scale is determined by the baroclinic Rossby radius of deformation, which is proportional to the depth-averaged buoyancy frequency [5].

This study aims to quantify the vertical wave momentum flux of internal waves, incorporating the dependence of horizontal exchange coefficients on the phenomenon's scale via the "4/3" law and on the buoyancy frequency.

Problem formulation

In the Boussinesq approximation, this study examines free progressive internal waves in an unbounded sea of constant depth in the presence of a shear current [25–31]. Unlike previous models that assumed constant horizontal turbulent exchange coefficients [25–31], this work incorporates coefficients that vary with the vertical coordinate and the horizontal scale of the phenomenon. In the linear approximation, the amplitude and dispersion characteristics of internal waves are determined, while in the second-order approximation with respect to wave amplitude, the vertical wave momentum flux and Stokes drift velocity components are calculated [25–27].

The system of hydrodynamic equations governing wave perturbations is as follows [25–31]:

$$\frac{D\mathbf{u}}{Dt} + w\frac{d\mathbf{U}^0}{dz} = -\frac{1}{\bar{\rho}_0}\nabla P + \mathbf{g}\frac{\rho}{\bar{\rho}_0} + K\Delta_h\mathbf{u}\,, \tag{2}$$

$$\frac{D\rho}{Dt} + (\mathbf{u}\nabla)\rho_0 = M\Delta_h\rho\,, \tag{3}$$

$$div\mathbf{u} = 0\,, \tag{4}$$

where $\mathbf{u}(u,v,w)$ is the vector of wave-induced current velocity perturbations; *x*-axis of the Cartesian three-dimensional coordinate system is directed along the mean plane-parallel current; z-axis is directed opposite to the gravitational acceleration vector $\mathbf{g}$; $\mathbf{U^0}$ $\left(U_0(z), 0, 0\right)$ is the mean current velocity vector; ρ, P are the wave-induced density and pressure perturbations [25–39]; $\rho_0(z)$ is the unperturbed mean density; K, M are the coefficients of horizontal turbulent viscosity and diffusion; the action of the $\dfrac{D}{Dt}$ operator is defined as

[5] Belonenko, T.V. and Novoselova, E.V., 2019. [*A Method for Estimating the Baroclinic Rossby Deformation Radius: A Textbook*]. Saint Petersburg: SPbGU, 25 p. (in Russian).

$\frac{D}{Dt}=\frac{\partial}{\partial t}+\left((\mathbf{u}+\mathbf{U^0})\nabla\right)$ [25–39]. The boundary conditions are the "rigid lid" condition at the surface [22, 25–39] and the "free-slip" condition at the bottom [23, 25–31].

Linear approximation. The solutions of the linear approximation for a progressive wave have the following form [25–39]:

$$\{\mathbf{u}, P, \rho\} = A\{\mathbf{u}_1(z), P_1(z), \rho_1(z)\}\exp(i(kx-\omega t)) + \text{c.c.} \tag{5}$$

The substitution of formula (5) into the system (2)–(4) leads to a system of equations connecting the amplitude functions u_1, ρ_1, P_1 with w_1 [25–31]:

$$u_1 = \frac{i}{k}\frac{dw_1}{dz}, \qquad \Omega = \omega - k\cdot U_0,$$

$$\frac{P_1}{\rho_0(0)} = \frac{i}{k}\left[\frac{\Omega}{k}\frac{dw_1}{dz} + \frac{dU_0}{dz}w_1 + ikK\frac{dw_1}{dz}\right],$$

$$\rho_1 = \frac{w_1}{i\Omega - k^2M}\frac{d\rho_0}{dz}, \qquad v_1 = 0.$$

The amplitude function of the vertical velocity $w_1(z)$ satisfies the equation

$$\frac{d^2w_1}{dz^2} + a(z)\frac{dw_1}{dz} + b(z)w_1 = 0, \tag{6}$$

where

$$a(z) = \frac{ik^2}{\Omega + ik^2K}\cdot\frac{\partial K}{\partial z}, \quad b(z) = k^2\left[\frac{N^2}{\left(\Omega+ik^2M\right)\left(\Omega+ik^2K\right)} + \frac{\frac{d^2U_0}{dz^2}}{k\left(\Omega+ik^2K\right)} - 1\right],$$

$N^2 = -\frac{g}{\rho_0(0)}\frac{d\rho_0}{dz}$ is square of the buoyancy frequency [2] [25–39].

Boundary conditions for the function $w_1(z)$ [25–39]):

$$w_1(0) = w_1(-H) = 0. \tag{7}$$

Non-linear effects. The two components of the Stokes drift velocity are determined by the formulas [42, 25–31])

$$u_s = \frac{A_1A_1^*}{k}\left[\frac{1}{\omega}\frac{d}{dz}\left(w_1\frac{dw_1^*}{dz}\right) + \text{c.c.}\right], \tag{8}$$

$$w_s = iA_1A_1^*\left[\frac{1}{\omega} - \frac{1}{\omega^*}\right]\frac{d}{dz}\left(w_1 w_1^*\right), \tag{9}$$

where $A_1 = A\exp(\delta\omega \cdot t)$, $\delta\omega = \mathrm{Im}(\omega)$. Accounting for turbulent viscosity and diffusion leads to the wave frequency having a small imaginary part, and the vertical component of the Stokes drift velocity (9) is not equal to zero [25–31]. The influence of turbulent viscosity and diffusion on the horizontal component of the Stokes drift velocity (8) is considered below.

The vertical wave momentum flux $\overline{uw}$ is determined by formula [2] [25–27, 29, 31–35, 38]

$$\overline{uw} = \frac{i}{k}\left|A_1^2\right|\left(w_1^*\frac{dw_1}{dz} - w_1\frac{dw_1^*}{dz}\right). \tag{10}$$

The solution to the boundary value problem (6), (7) is comlex, therefore the momentum flux $\overline{uw}$ (10) is non-zero.

Calculation results and their analysis

To calculate the vertical wave momentum flux, this study uses field experiment data from the Strait of Gibraltar [43]. The experiment employed remote sensing and *in-situ* measurements to detect intense internal waves with amplitudes up to 16 m, identifying the first mode with a period of 14 minutes. Phase velocity estimates, derived from both measurement data and theoretical calculations using numerical solutions of the Taylor–Goldstein equation with current velocity and buoyancy frequency profiles (Fig. 1), demonstrate good agreement [43].

Previously, we assumed that $M = 1\ \mathrm{m^2 \cdot s^{-1}}$ on the considered scales. This study accounts for the dependence of the horizontal turbulent exchange coefficient on the phenomenon's scale and the Brunt–Väisälä frequency. Notably, stratification inhibits vertical exchange and suppresses small-scale turbulence but does not impede horizontal exchange; rather, it enhances it [44]. This is supported by field experiment data used to determine the horizontal turbulent exchange coefficients [45]. The resulting vertical profiles of this coefficient reveal an increase in the pycnocline region in the absence of cyclones. Therefore, we employ a modified Riley formula [44]:

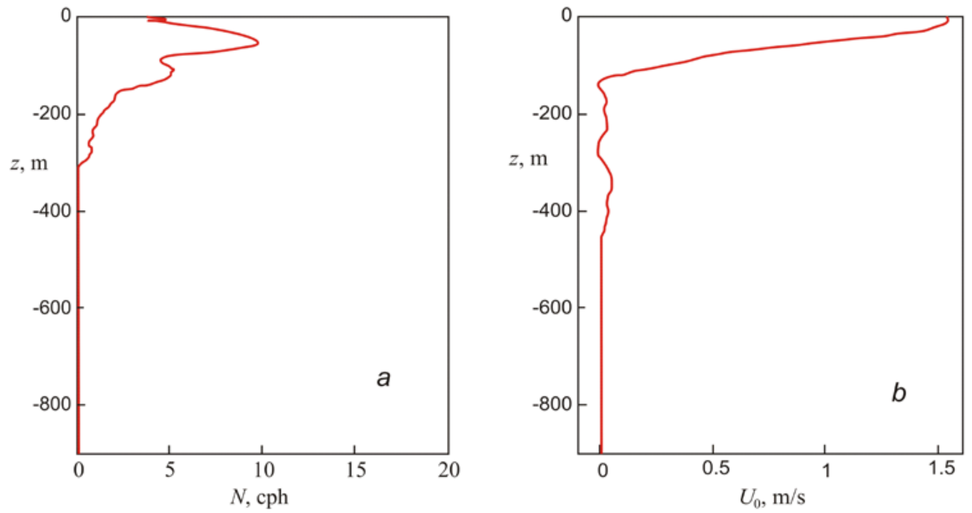
$$M = M_0\left(1 + \frac{N(z)}{N_0}\right), \tag{11}$$

where M_0 is the coefficient of horizontal turbulent exchange in a homogeneous fluid, which depends on the phenomenon's scale l according to the "4/3" law:

$$M_0 = c_1 \cdot l^{4/3}, \tag{12}$$

here the c_1 coefficient, based on measurement data from a large basin, is $c_1 = 0.01\ \mathrm{cm^{2/3} \cdot s^{-1}}$ (based on work [3]). From formulas (11) and (12), we determine the coefficient of horizontal turbulent exchange in the presence of stratification:

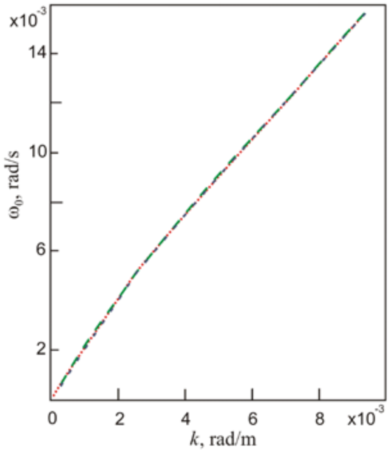
$$M = c_1\left(1+\frac{N(z)}{N_0}\right)\cdot l^{4/3} = C_1^{/}\cdot l^{4/3}. \qquad (13)$$



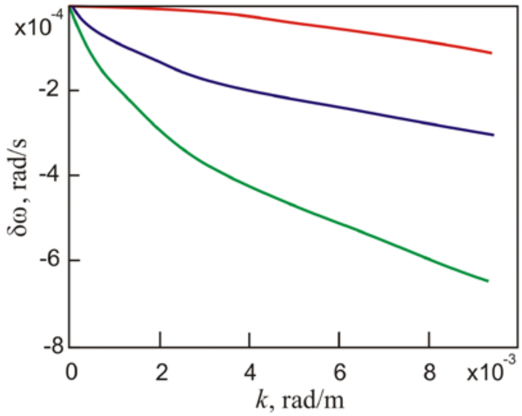
F i g. 1. Dependence of buoyancy frequency (*a*) and current velocity (*b*) on the vertical coordinate

Here, $N(z)$ represents the Brunt–Väisälä frequency in cycles per hour, with $N_0 = 5$ cph. This value of N_0 ensures that the empirical values of the c_1 coefficient (see work [3] and [46]) in the "4/3" law (1) align with the range of the $C_1^{/}(z)$ function. The scale of the phenomenon in formulas (1) and (13) is defined as the wavelength, i.e., $l = 2\pi / k$. The boundary value problem described by (6) and (7) is solved numerically using a second-order implicit Adams scheme at $K = 2M$ [26]. For a fixed real part of the wave frequency ω_0, the wavenumber and wave decay rate are determined using the shooting method [2] [25–31, 34–39]. The calculation results are compared for the turbulent exchange coefficient – fixed ($M = 1\,\mathrm{m^2{\cdot}s^{-1}}$) and dependent on the phenomenon's scale according to the "4/3" law, for both a constant value of $c_1 = 0.01\,\mathrm{cm^{2/3}\cdot s^{-1}}$ in formula (1) and for the case of dependence (formula (13)) of the exchange coefficient on stratification.

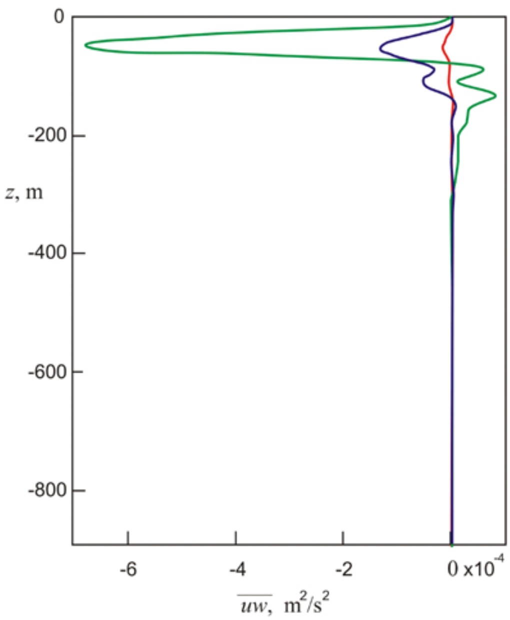
Fig. 2 presents the dispersion curves for the first mode in three cases. These dispersion curves are nearly identical, indicating that the real part of the wave frequency is largely insensitive to the dependence (1) of the exchange coefficient on the *l* scale and the dependence (13) of the exchange coefficient on the buoyancy frequency. However, the imaginary part of the wave frequency exhibits a notable dependence on the choice of the horizontal turbulent exchange coefficient (Fig. 3).



F i g. 2. Dispersion curves of the first mode of internal waves at three variants for choosing the coefficient of horizontal turbulent exchange



F i g. 3. Dependence of the wave attenuation decrement on wave number for three values of the horizontal exchange coefficient M: $1\,\mathrm{m^2{\cdot}s^{-1}}$ (red curve); calculated using formula (1) (blue curve) and using formula (13) (green curve)

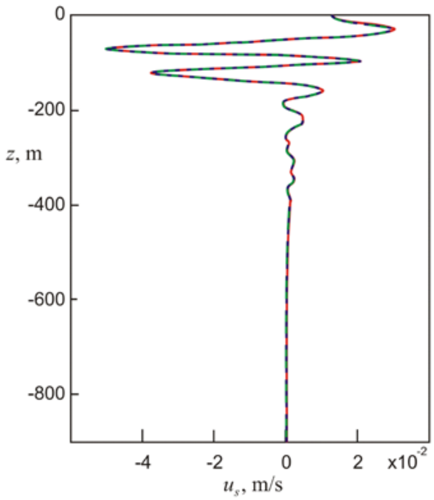


F i g. 4. Profiles of the vertical wave momentum flux $\overline{uw}$ at the exchange coefficients M: $1\,\mathrm{m^2{\cdot}s^{-1}}$ (red curve); calculated using formula (1) (blue curve) and using formula (13) (green curve)

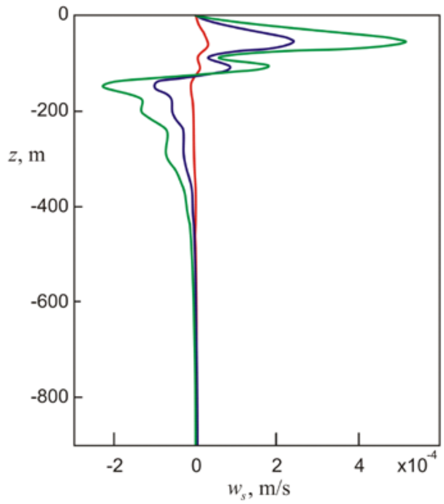
Hereafter, we assume that constant exchange coefficients correspond to variant 1, exchange coefficients dependent on the phenomenon's scale according to the "4/3" law correspond to variant 2, and variant 3 corresponds to formula (13), which accounts for the dependence of the horizontal turbulent exchange coefficient on the Brunt–Väisälä frequency. The wave decay rate for variant 2 (blue curve) in Fig. 3 is greater in magnitude than that for variant 1 (red curve) but smaller than that for variant 3 (green curve). The wavenumber of 14-minute internal waves of the lowest mode is $3.96 \cdot 10^{-3}$ rad/m [27, 30]. The normalization factor A_1 is determined based on the known maximum amplitude of vertical displacements [2] [25–39].

Similar calculations for the three variants of the turbulent exchange coefficient were conducted to evaluate the vertical wave momentum flux $\overline{uw}$ (10) for 14-minute internal waves of the first mode (Fig. 4).

In variant 2, where the turbulent exchange coefficient depends on the phenomenon's scale according to formula (1), the vertical wave momentum flux is noticeably greater in absolute value than in variant 1, when $M = 1\ \mathrm{m^2{\cdot}s^{-1}}$, but lower in magnitude than the momentum flux in variant 3. Calculations for the two components of the Stokes drift velocity were conducted similarly for the three variants of the turbulent exchange coefficient (Figs. 5, 6).



F i g. 5. Vertical distribution of horizontal component of the Stokes drift velocity



F i g. 6. Dependence of vertical component of the Stokes drift velocity on depth for three values of the horizontal exchange coefficient *M*: $1\ \mathrm{m^2{\cdot}s^{-1}}$ (red curve); calculated using formula (1) (blue curve) and using formula (13) (green curve)

The choice of the exchange coefficient has minimal influence on the horizontal component of the Stokes drift velocity (8). Fig. 6 presents the calculation results for the vertical component of the Stokes drift velocity (9). The magnitude of this velocity component in variant 1 is smaller than in variant 2, which, in turn, is smaller than in variant 3.

Conclusion

The vertical wave momentum flux of internal waves, when accounting for horizontal turbulent viscosity and diffusion, is non-zero. This results from the complex coefficients in the equation for the vertical velocity amplitude, rendering the solution to the boundary value problem (6), (7) complex. The wave frequency is also complex, with a small imaginary part representing the wave decay rate, which is determined during the solution of this boundary value problem. A phase shift different from $\pi/2$ exists between the components of the wave velocity perturbations, resulting in a non-zero vertical wave momentum flux. For a constant turbulent exchange coefficient $M = 1\ \mathrm{m^2{\cdot}s^{-1}}$, the vertical wave momentum flux is significantly smaller in magnitude than when the exchange coefficient depends on the phenomenon's scale according to formula (1). In the latter case, the momentum flux is noticeably smaller in absolute value than when the exchange coefficient is governed by formula (13), which accounts for stratification. The choice of the exchange coefficient has minimal impact on the dispersion curves; however, the wave decay rate is sensitive to this choice.

The wave decay rate for $M = 1\,\mathrm{m^2{\cdot}s^{-1}}$ is the smallest in magnitude. It increases when the exchange coefficient is determined by formula (1) and becomes the largest in absolute magnitude when calculated using formula (13) at a constant wavenumber. The horizontal component of the Stokes drift velocity is largely independent of the choice of the turbulent exchange coefficient, whereas the vertical component is significantly greater in absolute magnitude when the turbulent exchange coefficient depends on the phenomenon's scale according to formula (1), compared to the case of a constant exchange coefficient $M = 1\,\mathrm{m^2{\cdot}s^{-1}}$. The dependence (13), which accounts for the influence of stratification on the coefficient of vertical turbulent exchange, further increases the vertical component of the Stokes drift velocity. The vertical component of the Stokes drift velocity plays a significant role in the vertical transport of heat and salt.

REFERENCES

1. Ozmidov, R.V., 1965. [On Turbulent Exchange in a Stably Stratified Ocean]. *Izvestiya AN SSSR, Fizika Atmosfery i Okeana*, 1(8), pp. 853-860 (in Russian).

2. Wunsch, C. and Ferrari, R., 2004. Vertical Mixing, Energy, and the General Circulation of the Oceans. *Annual Review of Fluid Mechanics*, 36(1), pp. 281-314. https://doi.org/10.1146/annurev.fluid.36.050802.122121

3. Holford, J.M. and Linden, P.F., 1999. Turbulent Mixing in a Stratified Fluid. *Dynamics of Atmosphere and Oceans*, 30(2-4), pp. 173-198. https://doi.org/10.1016/S0377-0265(99)00025-1

4. Samodurov, A.S., Lubitsky, A.A. and Panteleev, N.A., 1995. Contribution of Breaking Internal Waves to Structure Formation, Energy Dissipation, and Vertical Diffusion in the Ocean. *Physical Oceanography*, 6(3), pp. 177-190. https://doi.org/10.1007/BF02197516

5. Podymov, O.I., Zatsepin, A.G. and Ostrovsky, A.G., 2017. Vertical Turbulent Exchange in the Black Sea Pycnocline and Its Relation to Water Dynamics. *Oceanology*, 57(4), pp. 492-504. https://doi.org/10.1134/S0001437017040142

6. Okhotnikov, I.N. and Panteleev, N.A., 1985. Shear Instability of Internal Waves and Vertical Exchange in the Ocean. *Morskoy Gidrofizicheskiy Zhurnal*, (3), pp. 13-20 (in Russian).

7. Desabies, Y. and Smith, W.K., 1982. Statistics of Richardson Number and Instability in Oceanic Internal Waves. *Journal of Physical Oceanography*, 12(11), pp. 1245-1269. https://doi.org/10.1175/1520-0485(1982)012<1245:SORNAI>2.0.CO;2

8. Zatsepin, A.G., Golenko, N.N., Korzh, A.O., Kremenetskii, V.V., Paka, V.T., Poyarkov, S.G. and Stunzhas, P.A., 2007. Influence of the Dynamics of Currents on the Hydrophysical Structure of the Waters and the Vertical Exchange in the Active Layer of the Black Sea. *Oceanology*, 47(3), pp. 301-312. https://doi.org/10.1134/S0001437007030022

9. Ivanov, A.V., Ostrovsky, L.A., Soustova, I.A. and Tsimring, L.S., 1983. Interaction of Internal Waves and Turbulence in the Ocean. *Dynamics of Atmosheres and Ocean*, 7(4), pp. 221-232. https://doi.org/10.1016/0377-0265(83)90006-4

10. Soustova, I.A., Troitskaya, Yu.I., Gladskikh, D.S., Mortikov, E.V. and Sergeev, D.A., 2020. A Simple Description of the Turbulent Transport in a Stratified Shear Flow as Applied to the Description of Thermohydrodynamics of Inland Water Bodies. *Izvestiya, Atmospheric and Oceanic Physics*, 56(6), pp. 603-612. https://doi.org/10.1134/S0001433820060109

11. Itsweire, E.C., Helland, K.N. and Van Atta, C.W., 1986. The Evolution of Grid-Generated Turbulence in a Stably Stratified Fluid. *Journal of Fluid Mechanics*, 162(2), pp. 299-338. https://doi.org/10.1017/S0022112086002069

12. Puig, P., Palanques, A., Guillén, J. and El Khatab, M., 2004. Role of Internal Waves in the Generation of Nepheloid Layers on the Northwestern Alboran Slope: Implications for Continental Margin Shaping. *Journal of Geophysical Research: Oceans*, 109(C9), C09011. https://doi.org/10.1029/2004JC002394

13. Kelly, R.E. and Maslowe, S.A., 1970. The Non-Linear Critical Layer in a Slightly Stratified Shear Flow. *Studies in Applied Mathematics*, 49(4), pp. 301-326.

14. Robinson, J.L., 1974. The Inviscid Nonlinear Instability of Parallel Shear Flows. *Journal of Fluid Mechanics*, 63(4), pp. 723-752.

15. Ostrovsky, L.A. and Zaborskikh, D.V., 1996. Damping of Internal Gravity Waves by Small-Scale Turbulence. *Journal of Physical Oceanography,* 26(3), pp. 388-397.

16. Druzhinin, O.A. and Ostrovsky, L.A., 2015. Dynamics of Turbulence under the Effect of Stratification and Internal Waves. *Nonlinear Processes in Geophysics*, 22(3), pp. 337-348. https://doi.org/10.5194/npg-22-337-2015

17. Badulin, S.I., Tsymring, L.Sh. and Shrira, V.A., 1983. [Capture and Vertical Focusing of Internal Waves in the Pycnocline by Horizontal Density Field and Current Heterogeneities]. *Doklady Akademii Nauk SSSR*, 273(2), pp. 459-463 (in Russian).

18. Bulatov, V.V. and Vladimirov, Yu.V., 2015. *Waves in Stratified Media*. Moscow: Nauka, 735 p. (in Russian).

19. Ivanov, V.A., Shul'ga, T.Ya., Bagaev, A.V., Medvedeva, A.V., Plastun, T.V., Verzhevskaia, L.V. and Svishcheva, I.A., 2019. Internal Waves on the Black Sea Shelf near the Heracles Peninsula: Modeling and Observation. *Physical Oceanography*, 26(4), pp. 288-304. https://doi.org/10.22449/1573-160X-2019-4-288-304

20. Borisenko, Yu.D., Voronovich, A.G., Leonov, A.I. and Miropolsky, Yu.Z., 1976. On the Theory of Nonstationary Weak Nonlinear Internal Waves in Stratified Fluid. *Izvestiya AN SSSR, Fizika Atmosfery i Okeana*, 12(3), pp. 293-301 (in Russian).

21. Grimshaw, R.H.J., 1977. The Modulation of an Internal Gravity-Wave Packet, and the Resonance with the Mean Motion. *Studies in Applied Mathematics*, 56(3), pp. 241-266. https://doi.org/10.1002/sapm1977563241

22. LeBlond, P.H. and Mysak, L.A., 1978. *Waves in the Ocean*. Elsevier Oceanography Series. 20Amsterdam – Oxford – New York: Elsevier Scientific Publishing Company, 602 p.

23. LeBlond, P.H., 1966. On the Damping of Internal Gravity Waves in a Continuously Stratified Ocean. *Journal of Fluid Mechanics*, 25(1), pp. 121-142. https://doi.org/10.1017/S0022112066000089

24. Ostrovsky, L.A. and Soustova, I.A., 1979. The Upper Mixed Layer of the Ocean as an Energy Sink of Internal Waves. *Okeanologiya*, 19(6), pp. 973-981 (in Russian).

25. Slepyshev, A.A., 2016. Vertical Momentum Transfer by Internal Waves When Eddy Viscosity and Diffusion Are Taken into Account. *Izvestiya, Atmospheric and Oceanic Physics*, 52(3), pp. 301-308. https://doi.org/10.1134/S0001433816030117

26. Slepyshev, A.A., 2022. Vertical Momentum Transfer by Internal Waves in a Shear Flow Taking into Account Turbulent Viscosity and Diffusion. *Izvestiya, Atmospheric and Oceanic Physics*, 58(5), pp. 433-439. https://doi.org/10.1134/S0001433822050115

27. Slepyshev, A.A. and Nosova, A.V., 2022. Vertical Transfer of Momentum by Internal Waves in the Western Part of the Mediterranean Sea. *Physical Oceanography*, 29(4), pp. 334-346. https://doi.org/10.22449/1573-160X-2022-4-334-346

28. Slepyshev, A.A., 2023. Generation of a Vertical Fine Structure by Internal Waves on the Sea Shelf. *Fluid Dynamics*, 58(3), pp. 413-426. https://doi.org/10.1134/S0015462822602200

29. Slepyshev, A.A., 2022. Vertical Momentum Transfer Induced by Internal Inertial-Gravity Waves in Flow with Account of Turbulent Viscosity and Diffusion. *Fluid Dynamics*, 57(2), pp. 183-192. https://doi.org/10.1134/S0015462822020094

30. Slepyshev, A.A. and Nosova, A.V., 2020. Generation of Vertical Fine Structure by the Internal Waves with the Regard for Turbulent Viscosity and Diffusion. *Physical Oceanography*, 27(1), pp. 3-17. https://doi.org/10.22449/1573-160X-2020-1-3-17

31. Slepyshev, A.A., 2015. Vertical Fluxes Induced by Weak-Nonlinear Internal Waves in a Baroclinic Flow. *Physical Oceanography*, (1), pp. 59-72. https://doi.org/10.22449/1573-160X-2015-1-59-72

32. Slepyshev, A.A. and Laktionova, N.V., 2019. Vertical Transport of Momentum by Internal Waves in a Shear Current. *Izvestiya, Atmospheric and Oceanic Physics*, 55(6), pp. 662-668. https://doi.org/10.1134/S0001433819060148

33. Slepyshev, A.A. and Schadt, M.A., 2024. Influence of the Nontraditional Approximation on Momentum Transfer by Internal Waves in a Shear Flow. *Izvestiya, Atmospheric and Oceanic Physics,* 60(5), pp. 515-522. https://doi.org/10.1134/S000143382470052X

34. Slepyshev, A.A., 2021. Vertical Transfer of Momentum by Inertia-Gravity Internal Waves on a Two-Dimensional Shear Flow. *Physical Oceanography*, 28(4), pp. 363-375. https://doi.org/10.22449/1573-160X-2021-4-363-375

35. Ankudinov, N.O. and Slepyshev, A.A., 2021. Vertical Momentum Transfer Induced by Internal Waves in a Two-Dimensional Flow. *Fluid Dynamics*, 56(3), pp. 343-352. https://doi.org/10.1134/S0015462821030022

36. Slepyshev, A.A. and Vorotnikov, D.I., 2017. Vertical Heat and Salt Fluxes Induced by Inertia-Gravity Internal Waves on Sea Shelf. *Izvestiya, Atmospheric and Oceanic Physics*, 53(4), pp. 467-475. https://doi.org/10.1134/S0001433817040119

37. Bagatinskii, V.A. and Slepyshev, A.A., 2016. Vertical Momentum Transfer by Weakly Nonlinear Inertia-Gravity Internal Waves. *Fluid Dynamics*, 51(5), pp. 595-605. https://doi.org/10.1134/S0015462816050033

38. Slepyshev, A.A. and Ankudinov, N.O., 2024. Generation of Vertical Fine Structure by Internal Waves on a Shear Flow. *Physical Oceanography*, 31(2), pp. 161-177.

39. Slepyshev, A.A. and Vorotnikov, D.I., 2019. Generation of Vertical Fine Structure by Internal Waves in a Shear Flow. *Open Journal of Fluid Dynamics*, 9, pp. 140-157. https://doi.org/10.4236/ojfd.201992010

40. Zhurbas, V. and Oh, I.S., 2003. Lateral Diffusivity and Lagrangian Scales in the Pacific Ocean as Derived from Drifter Data. *Journal of Geophysical Research: Oceans*, 108(C5), 3141. https://doi.org/10.1029/2002JC001596

41. Zhurbas, V.M., Zatsepin, A.G., Grigor'eva, Yu.V., Poyarkov, S.G., Eremeev, V.N., Kremenetsky, V.V., Motyzhev, S.V., Stanichny, S.V., Soloviev, D.M. [et al.], 2004. Water Circulation and Characteristics of Currents of Different Scales in the Upper Layer of the Black Sea from Drifter Data. *Oceanology*, 44(1), pp. 30-43.

42. Longuet-Higgins, M.S., 1969. On the Transport of Mass by Time-Varying Ocean Currents. *Deep Sea Research and Oceanographic Abstracts*, 16(5), pp. 431-447. https://doi.org/10.1016/0011-7471(69)90031-X

43. Watson, G., 1994. Internal Waves in a Stratified Shear Flow: The Strait of Gibraltar. *Journal of Physical Oceanography*, 24(2), pp. 509-517. https://doi.org/10.1175/1520-0485(1994)024<0509:IWIASS>2.0.CO;2

44. Riley, G., 1951. Parameters of Turbulence in the Sea. *Journal of Marine Research*, 10(3), pp. 267-287.

45. Permyakov, M.S., Tarkhova, T.I. and Sergienko, A.S., 2005. [Assessment of Horizontal Coefficients of Turbulent Exchange in the Northwestern Pacific Ocean]. *Researched in Russia*, 8, pp. 860-869 (in Russian).

46. Nemchenko, V.I., 1964. [Study of Horizontal Turbulent Diffusion in the Atlantic Ocean]. *Okeanologiya*, 4(5), pp. 805-808 (in Russian).

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